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J. Ambjørn
and R.A. Janik:

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Towards a diagrammatic derivation of the Veneziano-Yankielowicz-Taylor superpotential

J. Ambjørn [a,*] and *R.A. Janik* [b]

[a] The Niels Bohr Institute, Copenhagen University
Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark.
email: ambjorn@nbi.dk

[c] Institute of Physics, Jagellonian University,
Reymonta 4, PL 30-059 Krakow, Poland.
email: ufrjanik@if.uj.edu.pl

Abstract

We show how it is possible to integrate out chiral matter fields in $\mathcal{N} = 1$ supersymmetric theories and in this way derive in a simple diagrammatic way the $N_f S \log S - S \log \det X$ part of the Veneziano-Yankielowicz-Taylor superpotential.

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Introduction

The recent renewed interest in the calculation of the glueball superpotential via matrix models [1] has led to an understanding of how to extract the non-logarithmic part of these superpotentials by ordinary diagrammatic methods [2]. Just as the matrix models in the applications to non-critical strings and 2d quantum gravity were convenient tools for solving specific combinatorial problems: the summation over all "triangulated" worldsheets with given weights, we understand now that the matrix model in the Dijkgraaf-Vafa (DV) context is an effective way of summing a set of ordinary Feynman graphs which by the magic of supersymmetry can be combined in such a way that they have no space-time dependence.

However, we are still left without a simple diagrammatic derivation of the logarithmic part of the glueball superpotential, the so called Veneziano-Yankielowicz-Taylor superpotential. This effective Lagrangian was originally derived for a pure $\mathcal{N}=1$ $U(N_c)$ gauge theory by Veneziano and Yankielowicz [3] by anomaly matching and, by the same method, generalized to a $U(N_c)$ theory with N_f flavors in the fundamental representation by Taylor, Veneziano and Yankielowicz [4]. It is given by

$$W_{eff}^{VYT}(S,X) = W_{eff}^{VY}(S) + W_{eff}^{matter}(S,X) \tag{1}$$

where $W_{eff}^{VY}(S)$ is the pure gauge part

$$W_{eff}^{VY}(S) = -N_c S \log \frac{S}{\Lambda^3} \tag{2}$$

while $W_{eff}^{matter}(S,X)$ denotes the part coming from N_f flavors in the fundamental representation:

$$W_{eff}^{matter}(S,X) = N_f S \log \frac{S}{\Lambda^3} - S \log \frac{\det X}{\Lambda^2}. \tag{3}$$

In the above formulas S denotes the composite chiral superfield $\mathcal{W}_\alpha^2/32\pi^2$ and $X = \tilde{Q}Q$ is the $(N_f \times N_f)$ mesonic superfield, Q being the chiral matter field. In (2) and (3) Λ is an UV cut off. Usually this UV cut off is replaced by a renormalization group invariant scale Λ_M by use of the one-loop renormalization group:

$$\Lambda_M = \Lambda\, \mathrm{e}^{-\frac{8\pi^2}{(3N_c - N_f)\, g^2}}. \tag{4}$$

The beautiful derivation of (1)-(3) by anomaly matching has always been somewhat antagonizing since a clear diagrammatic understanding is missing. It is summarized in the following citation from [5]: "Its [i.e. (1)-(4)] only *raison d'etre* is the explicit realization of the anomalous and non-anomalous symmetries of SUSY gluodynamics ....".

In this letter we point out that there exists a simple diagrammatic derivation of (3). The derivation is inspired by diagrammatic techniques used in [2] and the observation that the DV-matrix models techniques could be extended to cover the case of superpotentials depending on mesonic superfields by considering the constrained (Wishart) matrix integrals [6]

$$\int DQD\tilde{Q}\ \delta(\tilde{Q}Q - X) = \frac{(2\pi)^{\frac{N(N+1)}{2}}}{\prod_{j=N-N_f+1}^{N}(j-1)!}\,(\det X)^{N-N_f} \tag{5}$$

and taking the large N limit.

Perturbative considerations

The matter contribution to the effective superpotential was shown in [2] to arise from the path integral

$$\int DQD\tilde{Q}\ \mathrm{e}^{\int d^4x d^2\theta\ \left(-\frac{1}{2}\tilde{Q}(\Box - i\mathcal{W}^\alpha\partial_\alpha)Q + W_{tree}(\tilde{Q},Q)\right)} \tag{6}$$

where $\mathcal{W}^\alpha$ is an external field and $\partial_\alpha \equiv \frac{\partial}{\partial\theta^\alpha}$. If the quarks are massive ($W_{tree} = m\tilde{Q}Q$) then the above path integral reduces to a functional determinant which can be easily evaluated using the Schwinger representation:

$$\frac{1}{2}\int_{\frac{1}{\Lambda}}^{\infty}\frac{\mathrm{d}s}{s}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int \mathrm{d}^2\pi_\alpha\ \exp\left(-s(p^2 + \mathcal{W}^\alpha\pi_\alpha + m)\right) \tag{7}$$

where we introduced an UV cut-off Λ. Due to fermionic integrations the result is

$$\frac{\mathcal{W}^2}{32\pi^2}\int_{\frac{1}{\Lambda}}^{\infty}\frac{ds}{s}e^{-ms} \tag{8}$$

which reduces for large Λ to

$$S\log\left(\frac{m}{\Lambda}\right) \tag{9}$$

At this stage one could integrate-in X to obtain (3). However, as "integrating-in" is in fact an assumption and we would like to obtain the desired result perturbatively, or more precisely: diagrammatically. To this end we impose the *superspace* constraint

$$X = \tilde{Q}Q \tag{10}$$

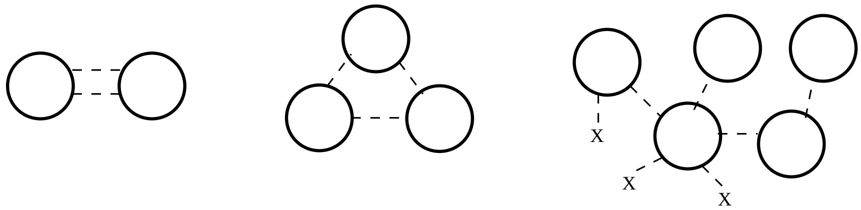

Figure 1: Only tree level graphs survive, i.e. we are left with the the graphs shown in fig. 1c).

at the level of the path integral (6). This is done by introducing a Lagrange multiplier chiral superfield α. Since the antichiral sector does not influence the chiral superpotentials, we will perform a trick analogous to [2] and introduce an antichiral partner $\bar{\alpha}$ with a tree level potential $M\bar{\alpha}^2$. Thus we have

$$\int \mathrm{d}^4x\mathrm{d}^4\theta\ \bar{\alpha}\alpha + \int \mathrm{d}^4x\mathrm{d}^2\theta\ M\bar{\alpha}^2. \tag{11}$$

The path integral w.r.t. $\bar{\alpha}$ is Gaussian and yields (c.f. [2])

$$-\frac{1}{2M}\int \mathrm{d}^4x\mathrm{d}^2\theta\ \alpha\Box\alpha. \tag{12}$$

The final path integral is

$$\int D\alpha D\tilde{Q}DQ\ \mathrm{e}^{\int d^4xd^2\theta\left(-\frac{1}{2}\tilde{Q}(\Box - i\mathcal{W}^\alpha\partial_\alpha)Q - \frac{1}{2}\alpha\Box\alpha - \alpha X + \alpha\tilde{Q}Q\right)}, \tag{13}$$

where we also took $W_{tree} = 0$ and fixed the auxiliary mass $M = 1$ (it will be clear from the arguments below that the result is independent of M).

This is no longer a free field theory, but nevertheless there are significant simplifications if we only want to extract the $\mathrm{tr}\,\mathcal{W}^2$ dependence. This implies that we must have two $\mathcal{W}$ insertions per $\tilde{Q}Q$ loop. The integrals over the fermionic momenta thus force all graphs which contain an α-line in a loop to vanish. Thus we are left with graphs coming from (13) which have the structure of $\tilde{Q}Q$ loops connected by at most one α propagator, and α propagators connected to the external field X as shown in fig. 1.

Moreover, if the field X contains a zero momentum component, which will generically be the case, the integrals will be dominated by this constant mode which forces the α propagators to be evaluated at zero momentum. Consequently we have to introduce an IR cut-off Λ_{IR}. Each 0-momentum α propagator will then just contribute a factor of $1/\Lambda_{IR}$. Thanks to the

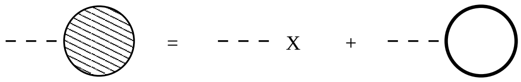

Figure 2: The Schwinger-Dyson equation for F.

above property we may find the full $\tilde{Q}Q$ propagator in terms of the α 1-point function which we will denote by F:

$$\frac{1}{p^2 + \mathcal{W}^\alpha \pi_\alpha + F}, \tag{14}$$

and the effective action will be given by the formula (7) with m substituted by F:

$$S \log \det \frac{F}{\Lambda} \tag{15}$$

It remains to determine F. The Schwinger-Dyson equation for F is (see fig. 2)

$$F = -\frac{1}{\Lambda_{IR}} X + \frac{1}{\Lambda_{IR}} \frac{S}{F} \tag{16}$$

where we used

$$\int_0^\infty \mathrm{d}s \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \int \mathrm{d}^2\pi_\alpha \, \mathrm{e}^{-s(p^2 + \mathcal{W}^\alpha \pi_\alpha + F)} = \frac{S}{F} \tag{17}$$

Eq. (16) is quadratic and has 2 solutions. Since the final result has to be IR finite, we will take the solution which has a finite limit as $\Lambda_{IR} \to 0$. Therefore

$$F = \frac{S}{X} \tag{18}$$

and by substituting this back in (15) one obtains the desired result:

$$S \log \det \frac{S X^{-1}}{\Lambda}, \tag{19}$$

or, in the case of N_f flavors:

$$N_f S \log \frac{S}{\Lambda^3} - S \log \det \frac{X}{\Lambda^2}. \tag{20}$$

Further examples

Exactly the same technique can be adapted to the theories studied in [7] where the matter effective superpotentials in terms of only mesonic fields are quite complex (see eqn. (1.1) in [7]) and follow from quite intricate physical analysis. However, as noted in [7] the superpotentials with both glueball fields and matter fields are simpler. The pure matter superpotentials can then be obtained by integrating out the glueball fields S_i.

The simplest case considered in [7] is a gauge theory with gauge group $SU(2)_1 \times SU(2)_2$, with a bifundamental matter field Q in the (2,2) representation. The natural gauge invariant matter superfield is

$$X = Q^2 \equiv \frac{1}{2} Q_{ab} Q_{cd} \varepsilon^{ac} \varepsilon^{bd}, \tag{21}$$

and the matter part of the superpotential $W_{eff}(S, X)$ is (eq. (4.19) in [7]):

$$(S_1 + S_2) \log \frac{S_1 + S_2}{X \Lambda} \tag{22}$$

We will now show that the expression (22) also follows from a diagrammatic reasoning.

Since for $SU(2)$ the fundamental and antifundamental representations are equivalent through $\tilde{Q}_a \equiv Q_{a'} \varepsilon^{a'a}$ the Lagrangian for the bifundamental fields takes the form:

$$Q_{a'b'} \varepsilon^{a'a} \varepsilon^{b'b} (\Box - i\mathcal{W}^{(1)\alpha}_{ac} \partial_\alpha - i\mathcal{W}^{(2)\alpha}_{bd} \partial_\alpha) Q_{cd} \tag{23}$$

Again we introduce a Lagrange multiplier superfield α enforcing the above constraint. We thus have

$$Q(\mathbf{C} \otimes \mathbf{C})(\Box - \mathcal{W}^{(1)\alpha} \otimes \mathbf{1}\pi_\alpha - \mathbf{1} \otimes \mathcal{W}^{(2)\alpha}\pi_\alpha + \frac{1}{2}\alpha)Q - \alpha X \tag{24}$$

where $\mathbf{C}^{ab} \equiv \varepsilon^{ab}$.

The analogue of formula (15) will then be

$$\frac{1}{2} 2(S_1 + S_2) \log \left(\frac{F}{2\Lambda} \right) \tag{25}$$

where the 1/2 comes from the fact that we are dealing with a real representation, while the 2 comes from performing the trace over the trivial factor in $(\mathcal{W}^{(1)} \otimes \mathbf{1})^2$. The Schwinger-Dyson equation for F will then have the form

$$F = -\frac{1}{\Lambda_{IR}} X + \frac{1}{\Lambda_{IR}} \frac{1}{2} \frac{2(S_1 + S_2)}{F/2} \tag{26}$$

hence

$$F = \frac{2(S_1 + S_2)}{X} \tag{27}$$

Inserting F into (25) reproduces precisely the nontrivial result (22).

Another example studied in [7] for the gauge group $SU(2)_1 \times SU(2)_2$ is matter $L\pm$ in the (1,2) representation. The classical D-flat direction is labeled by $Y = L_{\alpha+} L_{\beta-} \varepsilon^{\alpha\beta}$ and the matter contribution to W_{eff}^{VYT} was found in [7] to be:

$$S_2 \log \frac{S_2}{Y\Lambda}. \tag{28}$$

We can also reproduce this expression[1] by computing diagrammatically the contribution from the $L_\pm$ fields, starting with the Lagrangian

$$L(\mathbf{C} \otimes \mathbf{1})(\Box - \mathcal{W}^{(2)\alpha} \otimes \mathbf{1}\pi_\alpha + \alpha \mathbf{1} \otimes \mathbf{C}) L - \alpha Y, \tag{29}$$

where the second component in the tensor product is the flavor space.

Discussion

We have shown that it is possible to obtain the matter part of some generalized $W_{eff}^{VYT}(X, S)$ potentials by simple diagrammatic reasoning. It would be interesting to generalize the diagrammatic derivation to the gauge part of the Taylor-Veneziano-Yankielowicz superpotential. That would complete the diagrammatic derivation of the glueball superpotential.

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A perturbative window into nonperturbative physics, [hep-th/0208048].

[1] Up to a trivial rescaling of Λ. Note that in our approach the definition of the UV cut-off Λ (see e.g. (8)) is a matter of convention and may be modified.

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