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MOMENT FORMULAS FOR THE QUASI-NILPOTENT DT-OPERATOR

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ABSTRACT. Let T be the quasi-nilpotent DT-operator. By use of Voiculescu's amalgamated R-transform we compute the momets of $(T - \lambda 1)^*(T - \lambda 1)$ where $\lambda \in \mathbb{C}$, and the Brown-measure of $T + \sqrt{\epsilon}Y$, where Y is a circular element *-free from T for $\epsilon > 0$. Moreover we give a new proof of Śniady's formula for the moments $\tau(((T^*)^k T^k)^n)$ for $k, n \in \mathbb{N}$.

1. INTRODUCTION

The quasi-nilpotent DT-operator T was introduced by Dykema and the second author in [4]. It can be described as the limit in *-moments for $n \to \infty$, of random matrices of the form

$$T^{(n)} = \begin{pmatrix} 0 & t_{1,2} & \cdots & t_{1,n} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & t_{n-1,n} \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

where $\{\Re(t_{ij}), \Im(t_{ij})\}_{1 \le i < j \le n}$ is a set of n(n-1) independent identically distributed Gaussian random variables with mean 0 and variance $\frac{1}{2n}$. More precisely, T is an element in a finite von Neumann algebra, M, with a faithful normal tracial state, τ , such that for all $s_1, s_2, \ldots, s_k \in \{1, *\}$,

(1.1)
$$\tau(T^{s_1}T^{s_2}\cdots T^{s_k}) = \lim_{n\to\infty} \mathbb{E}[\operatorname{tr}_n((T^{(n)})^{s_1}(T^{(n)})^{s_2}\cdots (T^{(n)})^{s_k})],$$

where tr_n is the normalized trace on $M_n(\mathbb{C})$. Moreover the pair

 $(T, W^*(T))$ is uniquely determined up to *-isomorphism by (1.1). The quasinilpotent DT-operator can be realized as an element in the free group factor, $L(\mathbb{F}_2)$, in the following way (cf. [4, Sect. 4]): Let (D_0, X) be a pair of free selfadjoint elements in a tracial W^* -probability space (M, τ) , such that $d\mu_{D_0}(t) = 1_{[0,1]}(t)dt$ and X is semi-circular distributed, i.e. $d\mu_X(t) = \frac{1}{2\pi}\sqrt{4-t^2}1_{[-2,2]}(t)dt$. Then $W^*(D_0, X) \simeq W^*(D_0) \star W^*(X) \simeq L(\mathbb{F}_2)$. Put

$$T_N = \sum_{j=1}^{2^N} p_{N,j} X q_{N,j}$$

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for N = 1, 2, ..., where

$$p_{N,j} = 1_{\left[\frac{j-1}{2^N}, \frac{j}{2^N}\right]}(D_0), \quad q_{N,j} = 1_{\left[\frac{j}{2^N}, 1\right]}(D_0),$$

for $j = 1, 2, ..., 2^N$. Then $(T_N)_{N=1}^{\infty}$ converges in norm to an operator $T \in W^*(D_0, X)$, and the *-moments of T are given by (1.1), i.e. T is a realization of the quasi-nilpotent DT-operator. In the notation of [4, Sect. 4], $T = \mathcal{UT}(X, \lambda)$, where $\lambda : L^{\infty}[0, 1] \to W^*(D_0)$ is the *-isomorphism given by $\lambda(f) = f(D_0)$ for $f \in L^{\infty}([0, 1])$. In the following we put $\mathcal{D} = W^*(D_0) \simeq L^{\infty}([0, 1])$ and let $E_{\mathcal{D}}$ denote the trace-preserving conditional expectation of $W^*(D_0, X)$ onto \mathcal{D} .

In this paper we apply Voiculescu's \mathcal{R} -transform with amalgamation to compute various *-moments of T and of operators closely related to T. First we compute in section 3 moments and the scalar valued \mathcal{R} -transform of $(T - \lambda 1)^*(T - \lambda 1)$ for $\lambda \in \mathbb{C}$. The specialized case of $\lambda = 0$ was treated in [4] by more complicated methods. In section 4 we consider the operator

$$T + \sqrt{\epsilon}Y,$$

where Y is a circular operator *-free from T and $\epsilon > 0$. By random matrix considerations it is easily seen, that if T_1 and T_2 are two quasi-nilpotent DT-operators, which are *-free with respect to amalgamation over the same diagonal, \mathcal{D} , then $T + \sqrt{\epsilon}Y$ has the same *-distribution as $S = \sqrt{a}T_1 + \sqrt{b}T_2$, when $a = 1 + \epsilon$ and $b = \epsilon$ (cf. [1]). We use this fact to prove, that the Brown measure of $T + \sqrt{\epsilon}Y$ is equal to the uniform distribution on the closed disc $\overline{B}(0, \log(1+\frac{1}{\epsilon})^{-\frac{1}{2}})$ in the complex plane. Moreover we show, that the spectrum of $T + \sqrt{\epsilon}Y$ is equal to this disc, and that $T + \sqrt{\epsilon}Y$ is not a DT-operator for any $\epsilon > 0$.

In [4] it was conjectured, that

(1.2)
$$\tau(((T^*)^k T^k)^n) = \frac{n^{nk}}{(nk+1)!}$$

for $n, k \in \mathbb{N}$. This formula was proved by Sniady in [9]. Sniady's proof of (1.2) is based on Speicher's combinatorial approach to free probability with amalgamation from [11]. The key step in the proof of (1.2) was to establish a recursion formula for the \mathcal{D} -valued moments,

(1.3)
$$E_{\mathcal{D}}\left(\left((T^*)^k T^k\right)^n\right)$$

for each fixed $k \in \mathbb{N}$. Śniady's recursion formula for the \mathcal{D} -valued moments (1.3), was later used by Dykema and the second author to prove, that

$$W^*(T) = W^*(D_0, X) \simeq L(\mathbb{F}_2)$$

and that T admits a one parameter family of non-trivial hyperinvariant subspaces (cf. [5]). In section 5 and section 6 of this paper we give a new proof of Śniady's recursion formula for the \mathcal{D} -valued moments (1.3), which at the same time gives a new proof of (1.2). The new proof is based on Voiculescu's \mathcal{R} -transform with respect to amalgamation over $M_{2k}(\mathcal{D})$, the algebra of $2k \times 2k$ matrices over \mathcal{D} .

2. Preliminaries

In this section we give a few preliminaries on amalgamated probability theory. Let \mathcal{A} be a unital Banach algebra, and let \mathcal{B} be a Banach-sub-algebra containing the unit of \mathcal{A} . Then a map, $E_{\mathcal{B}} : \mathcal{A} \to \mathcal{B}$, is a conditional expectation if

- (a) $E_{\mathcal{B}}$ is linear,
- (b) $E_{\mathcal{B}}$ preserves the unit i.e. $E_{\mathcal{B}}(1) = 1$
- (c) and $E_{\mathcal{B}}$ has the \mathcal{B} , \mathcal{B} bi-module property i.e. $E_{\mathcal{B}}(b_1ab_2) = b_1ab_2$ for all $b_1, b_2 \in \mathcal{B}$ and $a \in \mathcal{A}$.

If \mathcal{B} , \mathcal{A} and $E_{\mathcal{B}}$ are as above we say that $(\mathcal{B} \subset \mathcal{A}, E_{\mathcal{B}})$ is a \mathcal{B} -probability space. If $\phi : \mathcal{A} \to \mathbb{C}$ is a state on \mathcal{A} which respects $E_{\mathcal{B}}$, i.e. $\tau = \tau \circ E_{\mathcal{B}}$, we say that $(\mathcal{B} \subset \mathcal{A}, E_{\mathcal{B}})$ is compatible to the (non-amalgamated) free probability space (\mathcal{A}, ϕ) .

If $(\mathcal{B} \subset A, E_{\mathcal{B}})$ is a \mathcal{B} -probability space and $a \in \mathcal{A}$ is a fixed variable, we define the amalgamated Cauchy transform of a by

$$G_a(b) = E_{\mathcal{B}}((b-a)^{-1}).$$

for $b \in \mathcal{B}$ and $b - a \in \mathcal{B}_{inv}$. The Cauchy transform is 1-1 in $\{b \in \mathcal{B}_{inv} | \|b^{-1}\| < \epsilon\}$ for ϵ sufficiently small and Voiculescu's amalgamated \mathcal{R} -transform [13] is now defined for $a \in \mathcal{A}$ by

(2.1)
$$\mathfrak{R}_a(b) = G_a^{\langle -1 \rangle}(b) - b^{-1},$$

for b being an invertible element of \mathcal{B} suitably close to zero. It turns out that this definition coincides on invertible element with Speicher's definition of the amalgamated \mathcal{R} -transform (cf. [11, Th. 4.1.2] and [2]);

(2.2)
$$\Re_a(b) = \sum_{n=1}^{\infty} \kappa_n^{\mathcal{B}} (a \otimes_{\mathcal{B}} ba \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} ba).$$

We will need the following useful lemma for solving equations involving the amalgamated *R*-transform and Cauchy-transform.

Lemma 2.1. Let $(\mathcal{B} \subset \mathcal{A}, E_{\mathcal{B}})$ be a \mathcal{B} -probability space, and let $a \in \mathcal{A}$. Then there exists $\delta > 0$ such that if $b \in \mathcal{B}$ is invertible, $||b|| < \delta$, $|\mu| > \frac{1}{\delta}$ and

$$\mathfrak{R}^{\mathcal{B}}_{a}(b) + b^{-1} = \mu \mathbf{1}_{\mathcal{A}}$$

then $b = G_a^{\mathcal{B}}(\mu 1_{\mathcal{A}}).$

Proof. Let $\delta = \frac{1}{11||a||}$ and define $g_b(b) = G_a^{\mathcal{B}}(b^{-1})$. By [2, Prop. 2.3] we know that g_a maps $\mathcal{B}(0, \frac{1}{4||a||})$ bijectively onto a neighboorhood of zero containing $\mathcal{B}(0, \frac{1}{11||a||})$ and furthermore that

$$g_a^{\langle -1 \rangle} \left(\mathcal{B}(0, \frac{1}{11 ||a||})_{\mathrm{inv}} \right) \subseteq \mathcal{B}(0, \frac{2}{11 ||a||})_{\mathrm{inv}}.$$

By definition we know that

$$\mathcal{R}_a^{\mathcal{B}}(b) = G_a^{\mathcal{B}^{\langle -1 \rangle}}(b) + b^{-1} = \left(g_a^{\langle -1 \rangle}(b)\right)^{-1} + b^{-1}$$

so if $\mathcal{R}_a(b) + b^{-1} = \mu \mathbf{1}_{\mathcal{A}}$ then

$$\mu 1_{\mathcal{A}} = g^{\langle -1 \rangle}(b) - b^{-1} + b^{-1} = \left(g_a^{\langle -1 \rangle}(b)\right)^{-1}$$

and thus

(2.3)
$$g_a^{\langle -1\rangle}(b) = \frac{1}{\mu} \mathbf{1}_{\mathcal{A}}$$

If $|\mu| > \frac{1}{\delta}$ then especially $\frac{1}{|\mu|} < \frac{1}{4||a||}$ so $\frac{1}{\mu} \mathbf{1}_{\mathcal{A}}$ is in the bijective domain of g_a , so applying g_a on both sides of (2.3) we get exactly

$$G_a^{\mathcal{B}}(\mu 1_{\mathcal{A}}) = g_a(\frac{1}{\mu} 1_{\mathcal{A}}) = b$$

since also $\|b\| < \frac{1}{11\|a\|}$.

If $a \in \mathcal{A}$ is a random variable in the \mathcal{B} -probability space $(\mathcal{B} \subset \mathcal{A}, E_{\mathcal{B}})$, then following Speicher we define a to be \mathcal{B} -Gaussian [11, Def 4.2.3] if only \mathcal{B} -cumulants of length 2 survive. From (2.2) it follows that in this case the \mathcal{R} -transform has a particularly simple form, namely,

(2.4)
$$\Re_a(b) = \kappa_2^{\mathcal{B}}(a \otimes_{\mathcal{B}} ba) = E_{\mathcal{B}}(aba)$$

In the following theorem (which is probably not a new one we just could not find a proper reference) concerning cumulants we have adopted the notation of Speicher from [11].

Lemma 2.2. Let $N \in \mathbb{N}$ and let $(\mathcal{B} \subset \mathcal{A}, E_{\mathcal{B}})$ be a \mathcal{B} -probability space. Then $(M_N(\mathcal{B}) \subset M_N(\mathcal{A}), E_{M_n(\mathcal{B})})$ is a $M_N(\mathcal{B})$ -probability space with cumulants determined by the following formula:

$$\kappa_n^{M_N(\mathcal{B})}((m_1 \otimes a_1) \otimes_{M_N(\mathcal{B})} \cdots \otimes_{M_N(\mathcal{B})} (m_n \otimes a_n)) = (m_1 \cdots m_n) \otimes \kappa_n^{\mathcal{B}}(a_1 \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} a_n)$$

when $m_1, \ldots, m_n \in M_N(\mathbb{C})$ and $a_1, \ldots, a_n \in \mathcal{A}$.

We have of course made the identification $M_N(\mathcal{A}) \cong M_N(\mathbb{C}) \otimes \mathcal{A}$.

Proof. Since $M_N(\mathbb{C}) \subset M_N(\mathcal{B})$ we observe that

$$\kappa_n^{M_N(\mathfrak{B})}((m_1 \otimes a_1) \otimes_{M_N(\mathfrak{B})} \cdots \otimes_{M_N(\mathfrak{B})} (m_n \otimes a_n)) = ((m_1 \cdots m_n) \otimes 1) \cdot \kappa_n^{M_N(\mathfrak{B})}((1 \otimes a_1) \otimes_{M_N(\mathfrak{B})} \cdots \otimes_{M_N(\mathfrak{B})} (1 \otimes a_n)).$$

To finish the proof we claim that

(2.5)
$$\kappa_n^{M_N(\mathcal{B})}((1 \otimes a_1) \otimes_{M_N(\mathcal{B})} \cdots \otimes_{M_N(\mathcal{B})} (1 \otimes a_n))$$

= $1 \otimes \kappa_n^{\mathcal{B}}(a_1 \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} a_n).$

The case n = 1 is obvious since

$$1_N \otimes \kappa_1^{\mathcal{B}}(a_1) = 1_N \otimes E_{\mathcal{B}}(a_1) = E_{M_N(\mathcal{B})}(1 \otimes a_1) = \kappa_1^{M_N(\mathcal{B})}(1 \otimes a_1).$$

Now assume that the claim is true for 1, 2, ..., n-1. Then (2.5) has an obvious extension to noncrossing particular of length less than or equal to n-1. Hence

$$1_{N} \otimes \kappa_{n}^{\mathcal{B}}(a_{1} \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} a_{n})$$

$$= 1_{N} \otimes E_{\mathcal{B}}(a_{1} \cdots a_{n}) - \sum_{\pi \in NC(n), \pi \neq 1_{n}} 1 \otimes \kappa_{\pi}^{\mathcal{B}}(a_{1} \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} a_{n})$$

$$= E_{M_{N}(\mathcal{B})}(1 \otimes_{M_{N}(\mathcal{B})} a_{1} \cdots a_{n})$$

$$- \sum_{\pi \in NC(n), \pi \neq 1_{n}} \kappa_{\pi}^{M_{N}(\mathcal{B})}((1 \otimes a_{1}) \otimes_{M_{N}(\mathcal{B})} \cdots \otimes_{M_{N}(\mathcal{B})} (1 \otimes a_{n}))$$

$$= \kappa_{n}^{M_{N}(\mathcal{B})}((1 \otimes a_{1}) \otimes_{M_{N}(\mathcal{B})} \cdots \otimes_{M_{N}(\mathcal{B})} (1 \otimes a_{n})).$$

By induction this proves the lemma.

Assume that \mathcal{M} contains a pair (D_0, X) of τ -free selfadjoint elements such that $d\mu_{D_0}(t) = 1_{[0,1]}(t)dt$ and X is a semicircular distributed. Put $\mathcal{D} = W^*(D_0)$. Then $\lambda : L^{\infty}([0,1]) \to \mathcal{D}$ given by

$$\lambda(f) = f(D_0),$$

for $f \in L^{\infty}([0,1])$ is a *-isomorphism of $L^{\infty}([0,1])$ onto \mathcal{D} and

$$\tau \circ \lambda(f) = \int_0^1 f(t) dt, \quad \mathbf{f} \in \mathcal{L}^{\infty}([0,1]).$$

We will identify \mathcal{D} with $L^{\infty}([0,1])$ and thus consider elements of \mathcal{D} as functions. As explained in the introduction, we can realize the quasi-nilpotent DT-operator as the operator $T = \mathcal{UT}(X,\lambda)$ in $W^*(D_0,X) \simeq L(\mathbb{F}_2)$.

Define for $f \in \mathcal{D} \simeq L^{\infty}([0,1])$

(2.6)
$$(L^*(f))(x) := \int_0^x f(t) dt$$
 and $(L(f))(x) := \int_x^1 f(t) dt$.

From the appendix of [5] it follows that (T, T^*) is a \mathcal{D} -Gaussian pair and that the covariances of (T, T^*) are given by the following lemma

Lemma 2.3. [5, Appendix] Let $f \in \mathcal{D}$. Then

$$E_{\mathcal{D}}(TfT^*) = L(f)$$
 and $E_{\mathcal{D}}(T^*fT) = L^*(f)$

and $E_{\mathcal{D}}(TfT) = E_{\mathcal{D}}(T^*fT^*) = 0.$

3. Moments and \mathcal{R} -transform of $(T - \lambda 1)^*(T - \lambda 1)$

Let T be the quasi-diagonal DT-operator and define

$$\tilde{T} = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}.$$

Since (T, T^*) is a \mathcal{D} -Gaussian pair, it follows from lemma 2.2, that cumulants of the form

$$\kappa_n^{M_2(\mathcal{D})}((m_1\otimes a_1)\otimes_{M_2(\mathcal{D})}\cdots\otimes_{M_2(\mathcal{D})}(m_n\otimes a_n))$$

vanishes when $n \neq 2$, $m_1, m_2, \ldots, m_n \in M_2(\mathbb{C})$ and $a_1, a_2, \ldots, a_n \in \{T, T^*\}$. Hence by the linearity of $\kappa_n^{M_2(\mathcal{D})}$,

$$\kappa_n^{M_2(\mathcal{D})}(\tilde{T}\otimes_{M_2(\mathcal{D})}\tilde{T}\otimes_{M_2(\mathcal{D})}\cdots\otimes_{M_2(\mathcal{D})}\tilde{T})=0$$

when $n \neq 2$, i.e. \tilde{T} is a $M_2(\mathcal{D})$ -Gaussian element in $M_2(\mathcal{M})$ under the conditional expectation $E_{M_2(\mathcal{D})} : M_2(\mathcal{M}) \to M_2(\mathcal{D})$ given by

$$E_{M_2(\mathcal{D})}: \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \longmapsto \begin{pmatrix} E_{\mathcal{D}}(a_{11}) & E_{\mathcal{D}}(a_{12}) \\ E_{\mathcal{D}}(a_{21}) & E_{\mathcal{D}}(a_{22}) \end{pmatrix}.$$

Since \tilde{T} is $M_2(\mathcal{D})$ -Gaussian the \mathcal{R} -transform of \tilde{T} is by (2.4) the linear mapping $M_2(\mathcal{D}) \to M_2(\mathcal{D})$ given by

$$\begin{aligned} \mathcal{R}_{\tilde{T}}^{M_{2}(\mathcal{D})}(z) &= E_{M_{2}(\mathcal{D})}(\tilde{T}z\tilde{T}) \\ &= E_{M_{2}(\mathcal{D})} \left(\begin{pmatrix} 0 & T^{*} \\ T & 0 \end{pmatrix} \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \begin{pmatrix} 0 & T^{*} \\ T & 0 \end{pmatrix} \right) \\ &= E_{M_{2}(\mathcal{D})} \left(\begin{pmatrix} T^{*}z_{22}T & 0 \\ 0 & Tz_{11}T^{*} \end{pmatrix} \right) \\ &= \begin{pmatrix} E_{\mathcal{D}}(T^{*}z_{22}T) & 0 \\ 0 & E_{\mathcal{D}}(Tz_{11}T^{*}) \end{pmatrix} \\ &= \begin{pmatrix} L^{*}(z_{22}) & 0 \\ 0 & L(z_{11}) \end{pmatrix}. \end{aligned}$$

For $\lambda \in \mathbb{C}$, we put $T_{\lambda} - T_{\lambda}1$ and define

$$\tilde{T}_{\lambda} = \begin{pmatrix} 0 & T_{\lambda}^* \\ T_{\lambda} & 0 \end{pmatrix} = \tilde{T} - \begin{pmatrix} 0 & \overline{\lambda}1 \\ \lambda1 & 0 \end{pmatrix}$$

Since $\begin{pmatrix} 0 & \overline{\lambda} \\ \lambda 1 & 0 \end{pmatrix} \in M_2(\mathcal{D})$ we have by $M_2(\mathcal{D})$ -freeness that the \mathcal{R} -transform is additive [11, Th. 4.1.22] i.e.

$$\mathcal{R}_{\tilde{T}_{\lambda}}^{M_{2}(\mathcal{D})}(z) = \mathcal{R}_{\tilde{T}}^{M_{2}(\mathcal{D})} - \begin{pmatrix} 0 & \overline{\lambda}1\\\lambda 1 & 0 \end{pmatrix} = \begin{pmatrix} L^{*}(z_{22}) & -\overline{\lambda}1\\-\lambda 1 & L(z_{11}) \end{pmatrix}$$

One easily checks, that if $\delta \in \mathbb{C}$, $\delta \neq 0$, $\delta \neq -\frac{1}{|\lambda|^2}$ and $\mu \in \mathbb{C}$ is one of the two solutions to

$$\mu^2 = \frac{\mathrm{e}^{\sigma}}{\sigma} (1 + |\lambda|^2 \sigma),$$

then

(3.1)
$$\begin{cases} z_{11} = \mu \sigma e^{\sigma(x-1)} \\ z_{12} = -\overline{\lambda} \sigma \\ z_{21} = -\lambda \sigma \\ z_{22} = \mu \sigma e^{-\sigma x} \end{cases}$$

is a solution to

$$\mathcal{R}^{M_2(\mathcal{D})}_{\tilde{T}_{\lambda}}(z) + z^{-1} = \mu \mathbb{1}_2.$$

Here x is the variable for the function in $\mathcal{D} = L^{\infty}([0, 1])$. In particular z_{12} and z_{21} are constant operators. If $\sigma \to 0$ then $|\mu| \to \infty$ and $||z|| \to 0$, so by lemma 2.1 there exists $\rho > 0$ such that $|\sigma| < \rho$ implies

$$G_{\tilde{T}_{\lambda}}^{M_{2}(\mathcal{D})}(\mu 1_{2}) = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix},$$

where $(z_{ij})_{i,j\in\{1,2\}}$ is given by (3.1) and

$$\mu = \pm \sqrt{\frac{\mathrm{e}^{\sigma}}{\sigma}(1+|\lambda|^2\sigma)}.$$

On the other hand the Cauchy-transform of \tilde{T} in $\mu 1_2$ is

$$\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} = G_{\tilde{T}_{\lambda}}^{M_{2}(\mathcal{D})}(\mu 1_{2})$$

$$= E_{M_{2}(\mathcal{D})} \left(\left(\begin{pmatrix} \mu 1 & 0 \\ 0 & \mu 1 \end{pmatrix} - \begin{pmatrix} 0 & T_{\lambda}^{*} \\ T_{\lambda} & 0 \end{pmatrix} \right)^{-1} \right)$$

$$= E_{M_{2}(\mathcal{D})} \left(\begin{pmatrix} \mu 1 & -T_{\lambda}^{*} \\ -T_{\lambda} & \mu 1 \end{pmatrix}^{-1} \right)$$

$$= E_{M_{2}(\mathcal{D})} \left(\begin{pmatrix} \mu (\mu^{2} 1 - T_{\lambda}^{*} T_{\lambda})^{-1} & T_{\lambda}^{*} (\mu^{2} 1 - T_{\lambda} T_{\lambda}^{*})^{-1} \\ T_{\lambda} (\mu^{2} 1 - T_{\lambda}^{*} T_{\lambda})^{-1} & \mu (\mu^{2} 1 - T_{\lambda} T_{\lambda}^{*})^{-1} \end{pmatrix} \right).$$

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Thus

(3.2)
$$\begin{cases} z_{11} = \mu E_{\mathcal{D}}((\mu^2 1 - T_{\lambda}^* T_{\lambda})^{-1}) \\ z_{12} = E_{\mathcal{D}}(T_{\lambda}^* (\mu^2 1 - T_{\lambda} T_{\lambda}^*)^{-1}) \\ z_{21} = E_{\mathcal{D}}(T_{\lambda} (\mu^2 1 - T_{\lambda}^* T_{\lambda})^{-1}) \\ z_{22} = \mu E_{\mathcal{D}}((\mu^2 1 - T_{\lambda} T_{\lambda}^*)^{-1}) \end{cases}$$

Combining (3.1) and (3.2) we have

(3.3)
$$\begin{cases} E_{\mathcal{D}}((\mu^{2}1 - T_{\lambda}^{*}T_{\lambda})^{-1}) = \sigma e^{\sigma(x-1)} \\ E_{\mathcal{D}}(T_{\lambda}^{*}(\mu^{2}1 - T_{\lambda}T_{\lambda}^{*})^{-1}) = -\overline{\lambda}\sigma \\ E_{\mathcal{D}}(T_{\lambda}(\mu^{2}1 - T_{\lambda}^{*}T_{\lambda})^{-1}) = -\lambda\sigma \\ E_{\mathcal{D}}((\mu^{2}1 - T_{\lambda}T_{\lambda}^{*})^{-1}) = \sigma e^{-\sigma x} \end{cases}$$

We can now compute the \mathcal{R} -transform of $T_{\lambda}^*T_{\lambda}$ (wrt. \mathbb{C}) from (3.3) and the defining equality for μ^2 .

$$\operatorname{tr}\left(\left(\frac{\mathrm{e}^{\sigma}}{\sigma}(1+|\lambda|^{2}\sigma)1-T_{\lambda}^{*}T_{\lambda}\right)^{-1}\right)=\int_{0}^{1}\sigma\mathrm{e}^{\sigma(x-1)}\mathrm{d}x$$
$$=\left[\mathrm{e}^{\sigma(x-1)}\right]_{0}^{1}=1-\mathrm{e}^{-\sigma}.$$

Thus

$$G_{T_{\lambda}^*T_{\lambda}}^{\mathbb{C}}\left(\frac{\mathrm{e}^{\sigma}}{\sigma}(1+|\lambda|^2\sigma)\right) = 1 - \mathrm{e}^{-\sigma}$$

i.e.

$$\mathfrak{R}^{\mathbb{C}}_{T^*_{\lambda}T_{\lambda}}(1-\mathrm{e}^{-\sigma}) = \frac{\mathrm{e}^{\sigma}}{\sigma}(1+|\lambda|^2\sigma) - \frac{1}{1-\mathrm{e}^{-\sigma}}$$

for σ in a neighboorhood of zero. Substituting $z = 1 - e^{-\sigma}$ we get $\sigma = -\log(1-z)$, so

$$\mathcal{R}^{\mathbb{C}}_{T^*_{\lambda}T_{\lambda}}(z) = -\frac{1}{(1-z)\log(1-z)}(1-|\lambda|^2\log(1-z)) - \frac{1}{z}.$$

Hence we have proved the following extension of [4, Theorem 8.7(b)]:

Theorem 3.1. Let T be the quasinilpotent DT-operator. Let $\lambda \in \mathbb{C}$ and put $T_{\lambda} = T - \lambda 1$. Then

$$\mathcal{R}^{\mathbb{C}}_{T_{\lambda}^{*}T_{\lambda}}(z) = -\frac{1}{(1-z)\log(1-z)} - \frac{1}{z} + \frac{|\lambda|^{2}}{1-z}$$

for z in some neighborhood of 0.

We next determine the \mathcal{D} -valued (resp. \mathbb{C} -valued) moments of $T_{\lambda}^*T_{\lambda}$ for all $\lambda \in \mathbb{C}$. The special case $\lambda = 0$ was treated in [9, Theorem 5] (resp. [4, Theorem 8.7(a)]) by different methods.

Theorem 3.2. Let $\lambda \in \mathbb{C}$ and let T, T_{λ} be as in theorem 3.1

(a) Let Q_n be the sequence of polynomials on \mathbb{R} uniquely determined by the following recursion formula

(3.4)
$$\begin{cases} Q_0(x) = 1, \\ Q_{n+1}(x) = |\lambda|^2 Q_n(x+1) + \int_0^x Q_n(y+1) dy] & \text{for } n \ge 1. \end{cases}$$

Then

$$E_{\mathcal{D}}((T_{\lambda}^*T_{\lambda})^n)(x) = Q_n(x), \quad x \in [0,1], \ n \in \mathbb{N}.$$

(b)

$$\tau((T_{\lambda}^*T_{\lambda})^n) = \sum_{k=0}^n \frac{n^k}{(k+1)!} \binom{n}{k} |\lambda|^{2n-2k}, \quad n \in \mathbb{N}.$$

Proof. By (3.3), we have

(3.5)
$$E_{\mathcal{D}}\left(\left(\frac{\mathrm{e}^{\sigma}}{\sigma}(1+|\lambda|^{2}\sigma)1-T_{\lambda}^{*}T_{\lambda}\right)^{-1}\right)=\sigma\mathrm{e}^{\sigma(x-1)}$$

for $\sigma \in B(0, \rho) \setminus \{0\}$ for some $\rho > 0$. Put

$$\psi(\sigma) = \frac{\sigma}{\mathrm{e}^{\sigma}(1+|\lambda|^2\sigma)}, \quad \sigma \in \mathbb{C} \setminus \{-\frac{1}{|\lambda|^2}\}.$$

Since $\psi(0) = 0$ and $\psi'(0) = 1$, ψ has an analytic invers $\psi^{\langle -1 \rangle}$ defined in a neighborhood $B(0, \delta)$ of 0, and we can choose $\delta > 0$, such that $\psi^{\langle -1 \rangle}(B(0, \delta)) \subset B(0, \rho)$. By (3.5)

$$E_{\mathcal{D}}((\frac{1}{t}1 - T_{\lambda}^*T_{\lambda})^{-1}) = \psi^{\langle -1 \rangle}(t) \mathrm{e}^{\psi^{\langle -1 \rangle}(t)(x-1)}$$

for $t \in B(0, \delta) \setminus \{0\}$. By power series expansion of the left hand side, we get

(3.6)
$$\sum_{n=0}^{\infty} t^{n+1} E_{\mathcal{D}}((T_{\lambda}^* T_{\lambda})^n) = \psi^{\langle -1 \rangle}(t) \mathrm{e}^{\psi^{\langle -1 \rangle}(t)(x-1)}$$

for $t \in B(0, \delta')$, where $0 < \delta' \leq \delta$ and where the LHS of (3.6) is absolutely convergent in the Banach space $L^{\infty}([0, 1])$. Hence by Cauchy's integral formulas

(3.7)
$$E_{\mathcal{D}}((T_{\lambda}^*T_{\lambda})^n) = \frac{1}{2\pi i} \int_C \frac{\psi^{\langle -1 \rangle}(t) e^{\psi^{\langle -1 \rangle}(t)(x-1)}}{t^{n+2}} dt$$

as a Banach space integral in $L^{\infty}([0,1])$, where $C = \partial B(0,r)$ with positive orientation and $0 < r < \delta'$. For each fixed $x \in \mathbb{R}$

$$t \mapsto \psi^{\langle -1 \rangle}(t) \mathrm{e}^{\psi^{\langle -1 \rangle}(t)(x-1)}$$

is an analytic function in $B(0, \delta')$ which is 0 for t = 0. Hence the function has a power series expansion of the form

(3.8)
$$\psi^{\langle -1\rangle}(t) e^{\psi^{\langle -1\rangle}(t)(x-1)} = \sum_{n=0}^{\infty} Q_n(x) t^{n+1}$$

for $t \in B(0, \delta')$, where the numbers $(Q_n(x))_{n=0}^{\infty}$ are given by

(3.9)
$$Q_n(x) = \frac{1}{2\pi i} \int_C \frac{\psi^{\langle -1 \rangle}(t) e^{\psi^{\langle -1 \rangle}(t)(x-1)}}{t^{n+2}} dt.$$

In particular the Q_n 's are continuous functions of $x \in \mathbb{R}$. Substituting $\sigma = \psi(t)$ in (3.8) we get

$$\sum_{n=0}^{\infty} Q_n(x)\psi(\sigma)^{n+1} = \sigma e^{\sigma(x-1)}$$

for $\sigma \in B(0, \rho')$, where $\rho' \in (0, \rho)$. Put

$$\begin{cases} R_0(x) = 0\\ R_{n+1}(x) = |\lambda|^2 Q_n(x+1) + \int_0^x Q_n(x) dy, \quad n \ge 0. \end{cases}$$

Then

$$\begin{split} \sum_{n=0}^{\infty} R_n(x)\psi(\sigma)^{n+1} &= \psi(\sigma) \left(1 + \sum_{n=0}^{\infty} R_{n+1}(x)\psi(\sigma)^{n+1} \right) \\ &= \psi(\sigma) \left(1 + |\lambda|^2 \left(\sum_{n+0}^{\infty} Q_n(x+1) \right) + \int_0^x \left(\sum_{n+0}^{\infty} Q_n(y+1) \right) \mathrm{dy} \right) \\ &= \psi(\sigma) \left(1 + |\lambda|^2 \sigma \mathrm{e}^{\sigma x} + \int_0^x \sigma \mathrm{e}^{\sigma y} \mathrm{dy} \right) \\ &= \psi(\sigma) (|\lambda|^2 \sigma + 1) \mathrm{e}^{\sigma x} = \sigma \mathrm{e}^{\sigma(x-1)} = \sum_{n=0}^{\infty} Q_n(x)\psi(\sigma)^{n+1} \end{split}$$

for all $\sigma \in B(0, \rho')$. Since $\psi(B(0, \rho'))$ is an open neighborhood of 0 in \mathbb{C} , it follows that $R_n(x) = Q_n(x)$ for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$.

Hence $(Q_n(x))_{n=0}^{\infty}$ is the sequence of polynomials given by the recursive formula (3.4). Moreover by (3.7) and (3.9), $E_{\mathcal{D}}((T_{\lambda}^*T_{\lambda})^n) = Q_n$ as functions in $L^{\infty}([0,1])$. This proves (a).

(b) By (3.7), we have

$$\tau((T_{\lambda}^*T_{\lambda})^n) = \int_0^1 E_{\mathcal{D}}((T_{\lambda}^*T_{\lambda})^n) \mathrm{dx} = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}} \frac{1 - \mathrm{e}^{-\psi^{\langle -1 \rangle}(\mathrm{t})}}{\mathrm{t}^{\mathrm{n}+2}} \mathrm{dt}.$$

Note that $C' = \psi(C)$ is a positively oriented simple path around 0. Hence by the substitution $t = \psi(\sigma)$, we get

$$\tau((T_{\lambda}^{*}T_{\lambda})^{n}) = \frac{1}{2\pi i} \int_{C'} \frac{\psi'(\sigma)}{\psi(\sigma)^{n+2}} (1 - e^{-\sigma}) d\sigma$$

$$= \frac{1}{2\pi i} \int_{C'} \frac{1}{n+1} \frac{1}{\psi(\sigma)^{n+1}} \frac{d}{d\sigma} (1 - e^{-\sigma}) d\sigma$$

$$= \frac{1}{2\pi i (n+1)} \int_{C'} \frac{1}{\psi(\sigma)^{n+1}} e^{-\sigma} d\sigma$$

$$= \frac{1}{n+1} \left(\frac{1}{2\pi i} \int_{C'} \frac{e^{n\sigma} (1 + |\lambda|^{2} \sigma)^{n+1}}{\sigma^{n+1}} d\sigma \right)$$

$$= \frac{1}{n+1} \operatorname{Res} \left(\frac{e^{n\sigma} (1 + |\lambda|^{2} \sigma)^{n+1}}{\sigma^{n+1}}, 0 \right)$$

where the second equation is obtained by partial integration and the last equality is obtained by the Residue theorem.

The above Residue is equal to the coefficient of σ^n in the Power series expansion of

$$e^{n\sigma}(1+|\lambda|^2\sigma)^{-1} = \left(\sum_{k=0}^{\infty} \frac{(n\sigma)^k}{k!}\right) \left(\sum_{i=1}^{n+1} \binom{n+i}{i} (|\lambda|^2\sigma)^i\right).$$

Hence

$$\tau((T_{\lambda}^{*}T_{\lambda})^{n}) = \frac{1}{n+1} \sum_{k=0}^{n} \frac{n^{k}}{k!} \binom{n+1}{n-k} |\lambda|^{2(n-k)}$$
$$= \frac{1}{n+1} \sum_{k=0}^{n} \frac{n^{k}}{(k+1)!} \binom{n}{k} |\lambda|^{2n-2k}.$$

4. Spectrum and Brown-measure of $T + \sqrt{\epsilon}Y$

Let T be the quasinilpotent DT-operator and let Y be a circular operator *-free from T. In this section we will show, that

$$\sigma(T + \sqrt{\epsilon}Y) = \overline{B}\left(0, \frac{1}{\sqrt{\log(1 + \epsilon^{-1})}}\right)$$

and that the Brown-measure $\mu_{T+\sqrt{\epsilon}Y}$ is equal to the uniform distribution on $\overline{B}\left(0,\frac{1}{\sqrt{\log(1+\frac{1}{\epsilon})}}\right)$, i.e. it has constant density w.r.t. the Lebesque measure on this disk.

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Theorem 4.1. For every $\epsilon > 0$

(4.1)
$$\sigma(T + \sqrt{\epsilon}Y) = \overline{B}\left(0, \frac{1}{\sqrt{\log(1 + \epsilon^{-1})}}\right).$$

Proof. The result can be obtained by the method of Biane and Lehner [3, Section 5]. Let $a \in \mathbb{C} \setminus \{0\}$. Since $\sigma(T) = \{0\}$ we can write

$$a1 - (T + \sqrt{\epsilon})Y = \sqrt{\epsilon} \left(\frac{1}{\sqrt{\epsilon}} 1 - Y(a1 - T)^{-1}\right)(a1 - T).$$

Hence

(4.2)
$$a \notin \sigma(T + \sqrt{\epsilon}Y) \quad \text{iff} \quad \frac{1}{\sqrt{\epsilon}} \notin \sigma(Y(a1 - T)^{-1}).$$

Let Y = UH be the polar decomposition of Y. Then $Y(a1 - T)^{-1} = UH(a1 - T)^{-1}$, where U is *-free from $H(a1 - T)^{-1}$. Hence $Y(a1 - T)^{-1}$ is R-diagonal. Moreover, since $0 \notin \sigma(Y)$, $Y(a1 - T)^{-1}$ is not invertible, so by [7, Prop. 4.6.(ii)]

(4.3)
$$\sigma(Y(a1-T)^{-1}) = B(0, ||Y(a1-T)^{-1}||_2).$$

By *-freeness of Y and $(a1 - T)^{-1}$ we have

(4.4)
$$\|Y(a1-T)^{-1}\|_{2}^{2} = \|Y\|_{2}^{2} \|(a1-T)^{-1}\|_{2}^{2}$$
$$= \|(a1-T)^{-1}\|_{2}^{2} = \|\sum_{n=0}^{\infty} \frac{T^{n}}{a^{n+1}}\|_{2}^{2}$$

Applying now [4, lemma 7.2] to D = 1 and $\lambda = \frac{1}{a}$ and $\mu = \delta_0$, we get

$$\left\|\sum_{n=0}^{\infty} \frac{T^n}{a^n}\right\|_2^2 = |a|^2 \left(\exp\left(\frac{1}{|a|^2}\right) - 1\right)$$

Hence by (4.4)

$$||Y(a1-T)^{-1}||_2^2 = \exp\left(\frac{1}{|a|^2}\right) - 1.$$

Thus for $a \in \mathbb{C} \setminus \{0\}$ we get by (4.2) and (4.3)

$$a \notin \sigma(T + \sqrt{\epsilon}Y) \iff \frac{1}{\sqrt{\epsilon}} \notin \sigma\left(Y(a1 - T)^{-1}\right)$$
$$\iff \frac{1}{\sqrt{\epsilon}} > \exp\left(\frac{1}{|a|^2}\right) - 1 \iff |a| > \frac{1}{\sqrt{\log(1 + \frac{1}{\epsilon})}}$$

Hence $\sigma(T + \sqrt{\epsilon}Y) \cup \{0\} = \overline{B}\left(0, \frac{1}{\sqrt{\log(1+\frac{1}{\epsilon})}}\right)$. Since $\sigma(T + \sqrt{\epsilon}Y)$ is closed it follows that $\sigma(T + \sqrt{\epsilon}Y) = \overline{B}\left(0, \frac{1}{\sqrt{\log(1+\frac{1}{\epsilon})}}\right)$.

In order to compute the Brown measure of $T + \sqrt{\epsilon}Y$, we first observe that $T + \sqrt{\epsilon}Y$ has the same *-distribution as

$$S = \sqrt{a}T_1 + \sqrt{b}T_2^*$$

when T_1 and T_2 are two \mathcal{D} -free quasidiagonal operators and $a = 1 + \epsilon$ and $b = \epsilon$ [1]. We next compute the Brown measure of S for all values of $a, b \in (0, \infty)$.

Lemma 4.2. Let μ_Q be the Brown measure of an operator Q in a tracial W^* -probability space (M, tr). Let r > 0 and assume that $\mu_Q(\partial B(0, r)) = 0$. Then

$$\mu_Q(B(0,r)) = -\frac{1}{2\pi} \lim_{\alpha \to 0^+} \Im\left(\int_{\partial B(0,r)} \operatorname{tr}((\mathbf{Q}^*_{\lambda}\mathbf{Q}_{\lambda} + \alpha \mathbf{1})^{-1}\mathbf{Q}^*_{\lambda}) \mathrm{d}\lambda\right)$$

where $Q_{\lambda} = Q - \lambda 1$ for $\lambda \in \mathbb{C}$.

Proof. Let $\Delta : M \to [0, \infty)$ be the Fuglede-Kadison determinant on M, and put $L(\lambda) = \log \Delta(Q_{\lambda})$ and

$$L_{\alpha}(\lambda) = \log \Delta((Q_{\lambda}^*Q_{\lambda} + \alpha 1)^{1/2}) = \frac{1}{2} \operatorname{tr}(\log(Q_{\lambda}^*Q_{\lambda} + \alpha 1))$$

for $\lambda \in \mathbb{C}$.

Put $\lambda_1 = \Re \lambda$, $\lambda_2 = \Im \lambda$ and let $\nabla^2 = \frac{\partial^2}{\partial \lambda_1^2} + \frac{\partial^2}{\partial \lambda_2^2}$ denote the Laplace operator on \mathbb{C} . Then by [6, Section 2] $\nabla^2 L_{\alpha} \ge 0$ and for each $\alpha > 0$, the measure

(4.5)
$$\mu_{\alpha} = \frac{1}{2\pi} \nabla^2 L_{\alpha}(\lambda) \mathrm{d}\lambda_1 \mathrm{d}\lambda_2$$

is a probability measure on \mathbb{C} . Moreover

(4.6)
$$\lim_{\alpha \to 0} \mu_{\alpha} = \mu$$

in the weak^{*} topology on Prob(\mathbb{C}). Also from [6, Section 2] the gradient $(\frac{\partial}{\partial\lambda_1}, \frac{\partial}{\partial\lambda_2})$ of L_{α} is given by

(4.7)
$$\frac{\partial}{\partial \lambda_1} L_{\alpha}(\lambda) = -\Re \left(\operatorname{tr}(Q_{\lambda}(Q_{\lambda}^* Q_{\lambda} + \alpha 1)^{-1}) \right)$$

(4.8)
$$\frac{\partial}{\partial \lambda_2} L_{\alpha}(\lambda) = -\Im \left(\operatorname{tr}(Q_{\lambda}(Q_{\lambda}^* Q_{\lambda} + \alpha 1)^{-1}) \right)$$

By (4.6)

$$\lim_{\alpha \to 0} \int_{\mathbb{C}} \phi \mathrm{d}\mu_{\alpha} = \int_{\mathbb{C}} \phi \mathrm{d}\mu$$

for all $\phi \in C_0(\mathbb{C})$. Since $1_{B(0,r)}$ is the limit of an increasing sequence $(\phi_n)_{n=1}^{\infty}$ of $C_0(\mathbb{C})$ -functions with $0 \leq \phi_n \leq 1$ for all $n \in \mathbb{N}$ it follows that

$$\mu_Q(B(0,r)) = \lim_{n \to \infty} \int_{\mathbb{C}} \phi_n d\mu_Q$$
$$= \lim_{n \to \infty} \left(\lim_{\alpha \to 0} \int_{\mathbb{C}} \phi_n d\mu_\alpha \right) \le \lim_{n \to \infty} \left(\liminf_{\alpha \to 0} \int_{\mathbb{C}} 1_{B(0,r)} d\mu_\alpha \right)$$
$$= \liminf_{\alpha \to 0} \mu_\alpha(B(0,r))$$

Writing $1_{\overline{B}(0,r)}$ as the limit of a decreasing sequence $(\psi_n)_{n=1}^{\infty}$ of $C_0(\mathbb{C})$ -functions, with $0 \leq \psi_n \leq 1$, one gets in the same way

$$\mu_Q(\overline{B}(0,r)) \ge \limsup_{\alpha \to 0} \mu_\alpha(\overline{B}(0,r))$$

Hence if $\mu_Q(\partial B(0, r)) = 0$ we have

$$\limsup_{\alpha \to 0} \mu_{\alpha}(B(0,r)) \le \mu_{Q}(B(0,r)) \le \liminf_{\alpha \to 0} \mu_{\alpha}(B(0,r)),$$

and therefore

$$\mu_Q(B(0,r)) = \lim_{\alpha \to 0} \mu_\alpha(B(0,r)).$$

Using (4.5) together with Green's theorem applied to the vector-field

$$(P_{\alpha}, Q_{\alpha}) = \left(-\frac{\partial L_{\alpha}}{\partial \lambda_2}, \frac{\partial L_{\alpha}}{\partial \lambda_1}\right)$$

we get

$$\mu_{\alpha}(B(0,r)) = \frac{1}{2\pi} \int_{B(0,r)} \nabla^2 L_{\alpha}(\lambda) d\lambda_1 d\lambda_2$$

$$= \frac{1}{2\pi} \int_{B(0,r)} \left(\frac{\partial Q_{\alpha}}{\partial \lambda_1} - \frac{\partial P_{\alpha}}{\partial \lambda_2} \right) d\lambda_1 d\lambda_2$$

$$= \frac{1}{2\pi} \int_{\partial B(0,r)} P_{\alpha} d\lambda_1 + Q_{\alpha} d\lambda_2$$

$$= \frac{1}{2\pi} \int_{\partial B(0,r)} -\frac{\partial L_{\alpha}}{\partial \lambda_2} d\lambda_1 + \frac{\partial L_{\alpha}}{\partial \lambda_1} d\lambda_2$$

$$= \Im \left(\frac{1}{2\pi} \int_{\partial B(0,r)} \left(\frac{\partial L_{\alpha}}{\partial \lambda_1} - i \frac{\partial L_{\alpha}}{\partial \lambda_2} \right) (d\lambda_1 + i d\lambda_2) \right)$$

By (4.7) and (4.8)

$$\frac{\partial L_{\alpha}}{\partial \lambda_1} - i \frac{\partial L_{\alpha}}{\partial \lambda_2} = -\overline{\operatorname{tr}(Q_{\lambda}(Q_{\lambda}^*Q_{\lambda} + \alpha 1)^{-1})} = -\operatorname{tr}((Q_{\lambda}^*Q_{\lambda} + \alpha 1)^{-1}Q_{\lambda}^*).$$

Hence

$$\mu_{\alpha}(B(0,r)) = -\Im\left(\frac{1}{2\pi}\int_{\partial B(0,r)} \operatorname{tr}((Q_{\lambda}^{*}Q_{\lambda} + \alpha 1)^{-1}Q_{\lambda}^{*})\mathrm{d}\lambda\right)$$

which completes the proof of the lemma.

Let $S = \sqrt{aT_1} + \sqrt{bT_2^*}$ with 0 < b < a. Since cS and S have the same *-distribution for all $c \in \mathbb{T}$, the Brown measure μ_S of S is rotation invariant (i.e. invariant under the transformation $z \mapsto cz$, $z \in \mathbb{C}$ when |c| = 1). Hence by lemma 4.2 we can compute μ_S , if we can determine

$$\operatorname{tr}((S_{\lambda}^*S_{\lambda} + \alpha 1)^{-1}S_{\lambda}^*)$$

for all $\lambda \in \mathbb{C}$, where $S_{\lambda} = S - \lambda 1$, and for all α in some interval of the form $(0, \alpha_0)$. This can be done by minor modifications of the methods used in section 3:

Put

$$\tilde{S}_{\lambda} = \begin{pmatrix} 0 & S_{\lambda}^* \\ S_{\lambda} & 0 \end{pmatrix}.$$

Then there exists a $\delta > 0$ (depending on a, b and γ) such that when $||z|| \leq \delta$ and $|\mu| > \frac{1}{\delta}$ the equality

(4.9)
$$\Re^{M_2(\mathcal{D})}_{\tilde{S}_{\lambda}}(z) + z^{-1} = \mu \mathbf{1}_2$$

implies that

(4.10)
$$z = G_{\tilde{S}_{\lambda}}^{M_{2}(\mathcal{D})}(\mu 1_{2}) \\ = (\mathrm{id} \otimes E_{\mathcal{D}}) \begin{pmatrix} \mu(\mu^{2}1 - S_{\lambda}^{*}S_{\lambda})^{-1} & S_{\lambda}^{*}(\mu^{2}1 - S_{\lambda}S_{\lambda}^{*})^{-1} \\ S_{\lambda}(\mu^{2}1 - S_{\lambda}^{*}S_{\lambda})^{-1} & \mu(\mu^{2}1 - S_{\lambda}S_{\lambda}^{*})^{-1} \end{pmatrix}.$$

Moreover, $\tilde{S} = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}$ is $M_2(\mathcal{D})$ -Gaussian by lemma 2.2 since (T_1, T_2^*, T_2, T_2^*) is a \mathcal{D} -Gaussian set. Hence for $z = (z_{ij})_{i,j=1}^2 \in M_2(\mathcal{D})$,

$$\mathfrak{R}^{M_2(\mathcal{D})}_{\tilde{S}}(z) = E_{M_2(\mathcal{D})}(\tilde{S}z\tilde{S}) = \begin{pmatrix} E_{\mathcal{D}}(S^*z_{22}S) & 0\\ 0 & E_{\mathcal{D}}(Sz_{11}S^*) \end{pmatrix}.$$

Using that (T_1, T_1^*) and (T_2, T_2^*) have the same \mathcal{D} -distribution as (T, T^*) and that (T_1, T_1^*) and (T_2, T_2^*) are two \mathcal{D} -free sets, we get

$$E_{\mathcal{D}}(S^* z_{22}S) = (aL^* + bL)(z_{22})$$
$$E_{\mathcal{D}}(Sz_{11}S^*) = (aL + bL^*)(z_{11}),$$

where $L(f) : x \mapsto \int_x^1 f(y) dy$ and $L^*(f) : x \mapsto \int_0^x f(y) dy$ for $f \in \mathcal{D}$. Since $\tilde{S}_{\lambda} = \tilde{S} - \begin{pmatrix} 0 & \overline{\lambda} 1\\ \lambda 1 & 0 \end{pmatrix}$ it follows that

$$\mathfrak{R}^{M_2(\mathfrak{D})}_{\tilde{S}}(z) = \begin{pmatrix} (aL+bL^*)(z_{22}) & -\overline{\lambda}1\\ \lambda 1 & (aL^*+bL)(z_{11}) \end{pmatrix}.$$

Thus (4.10) becomes

(4.11)
$$\begin{pmatrix} \mu 1 & 0 \\ 0 & \mu 1 \end{pmatrix} = \begin{pmatrix} (aL + bL^*)z_{22} & -\overline{\lambda}1 \\ \lambda 1 & (aL^* + bL)(z_{11}) \end{pmatrix} + \frac{1}{\det(z)} \begin{pmatrix} z_{22} & -z_{12} \\ -z_{21} & z_{11} \end{pmatrix}.$$

In analogy with section 3, we look for solutions $z_{ij} \in \mathcal{D} = L^{\infty}[0,1]$ of the form

(4.12)
$$\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} = \begin{pmatrix} c_{11} \exp(\sigma x) & c_{12} \\ c_{21} & c_{22} \exp(-\sigma x) \end{pmatrix},$$

where $\sigma \in \mathbb{C}$ and $c = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C})$. It is easy to check that (4.12) is a solution to (4.11) if the following 5 conditions are fulfilled:

$$\det(c) = \frac{\sigma}{a-b}$$

$$c_{11} = \frac{\sigma\mu}{ae^{\sigma}-b}$$

$$c_{12} = -\frac{\sigma\overline{\lambda}}{a-b}$$

$$c_{21} = -\frac{\sigma\lambda}{a-b}$$

$$c_{22} = \frac{\sigma\mu}{a-be^{-\sigma}}$$

The first of these conditions is consistent with the remaining 4 if and only if

$$\frac{(\sigma\mu)^2}{(a\mathrm{e}^{\sigma}-b)(a-b\mathrm{e}^{-\sigma})} - \frac{\sigma^2|\lambda|^2}{(a-b)^2} = \frac{\sigma}{a-b}$$

which is equivalent to

(4.13)
$$\mu^{2} = \frac{(ae^{\sigma} - b)(a - be^{-\sigma})(a - b + \sigma|\lambda|^{2})}{\sigma(a - b)^{2}}$$

Put

$$\sigma_0 := -\min\left\{\frac{a-b}{|\lambda|^2}, \log\left(\frac{a}{b}\right)\right\}.$$

Then for $\sigma_0 < \sigma < 0$, the right hand side of (4.13) is negative. Let in this case $\mu(\sigma)$ denote the solution to (4.13) with positive imaginary part, i.e.

(4.14)
$$\mu(\sigma) = i \frac{a e^{\sigma/2} - b e^{-\sigma/2}}{|\sigma|^{1/2} (a-b)} \sqrt{a-b+\sigma |\lambda|^2}$$

for $\sigma_0 < \sigma < 0$. Then with

$$c_{11} = \frac{\sigma\mu(\sigma)}{ae^{\sigma} - b} \qquad c_{12} = -\frac{\sigma\lambda}{a - b}$$
$$c_{21} = -\frac{\sigma\lambda}{a - b} \qquad c_{22} = \frac{\sigma\mu(\sigma)}{a - be^{-\sigma}}$$

the matrix $z(\sigma) = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$ given by (4.12) is a solution to

$$\mathfrak{R}^{M_2(\mathcal{D})}_{\tilde{S}_{\lambda}}(z(\sigma)) + z(\sigma)^{-1} = \mu \mathbb{1}_2.$$

By (4.14) $\lim_{\sigma\to 0^-} |\mu(\sigma)| = \infty$ and $\lim_{\sigma\to 0^-} |\sigma\mu(\sigma)| = 0$ and therefore

$$\lim_{\sigma \to 0^-} \|z(\sigma)\| = 0$$

Hence for some $\sigma_1 \in (\sigma_0, 0)$ we have $|\mu(\sigma)| > \frac{1}{\delta}$ and $||z(\sigma)|| > \delta$ when $\sigma \in (\sigma_1, 0)$ where $\delta > 0$ is the number described in connection with (4.9). Thus

(4.15)
$$z(\sigma) = G^{M_2(\mathcal{D})}_{\tilde{S}_{\lambda}}(\mu(\sigma)1_2)$$

for $\sigma \in (\sigma_1, 0)$. But since both $\sigma \mapsto z(\sigma)$ and $\sigma \mapsto \mu(\sigma)$ are analytic functions (of the real variable σ) it follows that (4.15) holds for all $\sigma \in (\sigma_0, 0)$. Note that $\sigma \mapsto -i\mu(\sigma)$ is a continuous strictly positive function on $(\sigma_0, 0)$, and

$$\lim_{\sigma \to 0^-} (-i\mu(\sigma)) = +\infty \qquad \lim_{\sigma \to \sigma_0^+} (-i\mu(\sigma)) = 0.$$

Hence for every fixed real number $\alpha > 0$ we can chose $\sigma \in (\sigma_0, 0)$, such that

$$-\mathrm{i}\mu(\sigma) = \sqrt{\alpha}.$$

Thus by (4.10) and (4.15)

$$E_{\mathcal{D}}(S_{\lambda}^{*}(-\alpha 1 - S_{\lambda}S_{\lambda}^{*})^{-1}) = z(\sigma)_{12} = -\frac{\sigma\lambda}{a-b}$$

which is a constant function in $L^{\infty}[0,1]$. Hence

$$\operatorname{tr}(S_{\lambda}^*(S_{\lambda}S_{\lambda}^* + \alpha 1)^{-1}) = \frac{\sigma\overline{\lambda}}{a-b}$$

from which

$$\int_{\partial B(0,r)} \operatorname{tr}(S_{\lambda}^*(S_{\lambda}S_{\lambda}^* + \alpha 1)^{-1}) \mathrm{d}\lambda = 2\pi \mathrm{i} \frac{\sigma r^2}{a-t}$$

when $\sigma_0 < \sigma < 0$, where as before $\sigma_0 = -\min\left\{\frac{a-b}{|\lambda|^2}, \log\left(\frac{a}{b}\right)\right\}$. Now $\alpha \to 0^+$ corresponds to $\sigma \to \sigma_0^+$. Hence

$$\lim_{\alpha \to 0^+} \left(-\frac{1}{2\pi} \Im \int_{\partial B(0,r)} \operatorname{tr}(S_{\lambda}^* (S_{\lambda} S_{\lambda}^* + \alpha 1)^{-1}) \mathrm{d}\lambda \right)$$
$$= -\frac{\sigma_0 r^2}{a-b} = +\min\left\{ 1, r^2 \frac{\log\left(\frac{a}{b}\right)}{a-b} \right\}.$$

Observe that $S_{\lambda}^*(S_{\lambda}S_{\lambda}^* + \alpha 1)^{-1} = (S_{\lambda}^*S_{\lambda} + \alpha 1)^{-1}S_{\lambda}^*$. Thus by lemma 4.2 we have for all but countably many r > 0, that

$$\mu_S(B(0,r)) = \min\left\{1, r^2 \frac{\log\left(\frac{a}{b}\right)}{a-b}\right\} = \begin{cases} r^2 \frac{\log\left(\frac{a}{b}\right)}{a-b}, & r \le \sqrt{\frac{a-b}{\log\left(\frac{a}{b}\right)}}\\ 1, & r > \sqrt{\frac{a-b}{\log\left(\frac{a}{b}\right)}} \end{cases}.$$

Since the right hand side is a continuous function of r, the formula actually holds for all r > 0. This together with the rotation invariance of μ_S shows, that μ_S is equal to the uniform distribution on

$$\overline{B}\Big(0,\sqrt{\frac{a-b}{\log\left(\frac{a}{b}\right)}}\Big),$$

i.e. has constant density $\frac{1}{\pi} \frac{\log(\frac{a}{b})}{a-b}$ on this ball, and vanishes outside the ball. Putting $a = 1 + \epsilon$ and $b = \epsilon$ we get in particular

Theorem 4.3. The Brown measure of $T + \sqrt{\epsilon}Y$ is equal to the uniform distribution on $\overline{B}\left(0, \frac{1}{\sqrt{\log(1+\epsilon^{-1})}}\right)$.

The Brown mesure of $T + \sqrt{\epsilon}Y$ can be used to give an upper bound of the microstate entropy of $T + \sqrt{\epsilon}Y$. By [8] we have for $S \in \mathcal{M}$

(4.16)
$$\chi(S) \leq \int_{\mathbb{C}} \int_{\mathbb{C}} \log |z_1 - z_2| \mathrm{d}\mu_S(z_1) \mathrm{d}\mu_S(z_2) + \frac{5}{4} + \log(\pi\sqrt{2\mathrm{od}_S})$$

where μ_S is the Brown measure of S on \mathbb{C} and od_S is the off-diagonality of S defined by

(4.17)
$$\operatorname{od}_{S} := \tau(SS^{*}) - \int_{\mathbb{C}} |z|^{2} \mathrm{d}\mu_{S}(z).$$

Lemma 4.4. For R > 0 we have

$$I := \int_{B(0,R)} \int_{B(0,R)} \log|z_1 - z_2| dz_1 dz_2 = \pi^2 (R^2 \log R - \frac{1}{4})$$

Proof. Polar substitution in I gives

$$I := 4\pi^2 \int_0^R \int_0^R \left(\frac{1}{2\pi} \int_0^{2\pi} \log|r - e^{i\theta}s| d\theta\right) r drs ds.$$

Let 0 < s < r. $z \mapsto \log |r - zs|$ is the real value of the complex holomorphic function $z \mapsto \text{Log}(r - zs)$, where Log is the principal branch of the complex logarithm, so $z \mapsto \log |r - zs|$ is a harmonic function in $B(0, \frac{r}{s})$. By the mean value property of harmonic functions

$$\frac{1}{2\pi} \int_0^{2\pi} \log |r - e^{i\theta}s| d\theta = \log(r),$$

so symmetry in r and s reduces I to

$$I := 4\pi^2 \int_0^R \int_0^R \max\{\log(r), \log(s)\} r dr s ds$$

= $8\pi^2 \int_0^R \left(\int_0^r \log(r) s ds \right) r dr$
= $4\pi^2 \int_0^R r^3 \log(r) dr = \pi^2 R^4 (\log(R) - \frac{1}{4}).$

Theorem 4.5.

(4.18)
$$\chi(T + \sqrt{\epsilon}Y) \le -\frac{1}{2}\log\left(\log(1 + \epsilon^{-1})\right) - \frac{1}{4} + \log\pi + \frac{1}{2}\log\left(1 + 2\epsilon - \frac{1}{\log(1 + \epsilon^{-1})}\right).$$

Proof. Let ν_R be the uniform distribution on $\overline{B}(0, R)$. Since ν_R has constant density $(\pi R^2)^{-1}$ on $\overline{B}(0, R)$, we have by lemma 4.4

$$\int_{\mathbb{C}} \int_{\mathbb{C}} \log |z_1 - z_2| d\nu_{\mathrm{R}}(z_1) d\nu_{\mathrm{R}}(z_2) = \log \mathrm{R} - \frac{1}{4}.$$

The Brown measure of $S = T + \sqrt{\epsilon}Y$ is $\mu_S = \nu_R$ with $R = \log(1 + \epsilon^{-1})^{-\frac{1}{2}}$, and

$$\mathrm{od}_{\mathrm{S}} = \frac{1}{2} + \epsilon - \int_{\mathbb{C}} |\mathbf{z}|^2 \mathrm{d}\nu_{\mathrm{R}} = \frac{1}{2} + \epsilon - \frac{\mathrm{R}^2}{2}$$

Hence by (4.16)

$$\chi(T + \sqrt{\epsilon}Y) \le \log R - \frac{1}{4} + \log \pi + \frac{1}{2}\log(1 + 2\epsilon - R^2).$$

This proves (4.18).

In [1] the first author proved that the microstate-free analog, $\delta_0^*(T)$, of the free entropy dimension is equal to 2. From Theorem 4.5 one gets only the trivial estimate of the free entropy dimension $\delta_0(T)$, namely

(4.19)
$$\delta_0(T) \le 2 + \lim_{\delta \to 0^+} \frac{\chi(T + \sqrt{2\delta Y})}{|\log \delta|} = 2.$$

If $T + \sqrt{\epsilon}Y$ was a DT-operator for all $\epsilon > 0$ then by [8] equality would hold in (4.18), and hence also in (4.19). In the rest of this section, we prove that unfortunately $T + \sqrt{\epsilon}Y$ is not a DT-operator for any $\epsilon > 0$.

If R = D + T is a $DT(\mu, 1)$ operator it follows from [4, lemma 7.2] that for $|\lambda| < ||R||^{-1}$,

$$\left\|\sum_{n=0}^{\infty}\lambda^{n}R^{n}\right\|_{2}^{2} = \frac{1}{|\lambda|^{2}}\left(\exp\left(\sum_{k,l=1}^{\infty}\lambda^{k+1}\overline{\lambda}^{l+1}M_{\mu}(k,l)-1\right)\right),$$

where $M_{\mu}(k, l) = \int_{\sigma(R)} z^k \overline{z}^l d\mu_R(z)$.

If thus μ_D is the uniform distribution on a disk with radius d then

$$M_{\mu_D}(k,l) = 0$$

when $k \neq l$ and

$$M_{\mu_D}(k,k) = \frac{1}{\pi d^2} \int_{B(0,d)} |z|^{2k} dz$$
$$= \frac{2\pi}{\pi d^2} \int_0^d r^{2k+1} dr = \frac{2}{d^2} \left[\frac{r^{2k+2}}{2k+2} \right]_0^r = \frac{d^{2k}}{k+1}$$

for $k \in \mathbb{N}$. Thus

(4.20)
$$\left\|\sum_{n=0}^{\infty} \lambda^{n} (D+T)^{n}\right\|_{2}^{2} = \frac{1}{|\lambda|^{2}} \left[\exp\left(\sum_{k=0}^{\infty} |\lambda|^{2(k+1)} \frac{d^{2k}}{k+1}\right) - 1\right]$$
$$= \frac{1}{|\lambda|^{2}} \exp\left(\frac{1}{d^{2}} \left(-\log(1-d^{2}|\lambda|^{2})\right)\right)$$
$$= \frac{1}{|\lambda|^{2}} \left[(1-d^{2}|\lambda|^{2})^{-\frac{1}{d^{2}}} - 1\right].$$

If instead D + cT is a $DT(\mu_D, c)$ operator with μ_D being the uniform distribution on a disc of radius d then

$$D + cT = c(D' + T)$$

where D' now has the uniform distribution on $B(0, \frac{d}{c})$, so from (4.20) we obtain

(4.21)
$$\left\|\sum_{n=0}^{\infty} \lambda^{n} (D+cT)^{n}\right\|_{2}^{2} = \left\|\sum_{n=0}^{\infty} (c\lambda)^{n} (D'+T)^{n}\right\|_{2}^{2} = \frac{1}{c^{2}|\lambda|^{2}} \left[\left(1-d^{2}|\lambda|^{2}\right)^{-\frac{c^{2}}{d^{2}}}-1\right].$$

Lemma 4.6. Let a > b > 0 and let $S = \sqrt{a}T_1 + \sqrt{b}T_2^*$ where T_1 and T_2 are two D-free quasidiagonal DT-operators. Then

$$\left\|\sum_{n=0}^{\infty} \lambda^n S^n\right\|_2^2 = \frac{1}{|\lambda|^2} \frac{e^{(a-b)|\lambda|^2} - 1}{a - be^{(a-b)|\lambda|^2}}, \quad |\lambda| < \frac{1}{\|S\|^2}.$$

Proof. Let $F_n(x) = E_{\mathcal{D}}((S^*)^n S^n)$ for $n \in \mathbb{N}$ and $x \in [0, 1]$. For $t < \frac{1}{\|S\|^2}$ define the \mathcal{D} -valued function

(4.22)
$$F(t,x) = \sum_{n=0}^{\infty} F_n(x)t^n.$$

By Speicher's cumulant formula we have by \mathcal{D} -Gaussianity of S that

$$F_n = E_{\mathcal{D}}((S^*)^n S^n) = \sum_{\pi \in \mathrm{NC}(2n)} \kappa_{\pi}^{\mathcal{D}} \left((S^*)^{\otimes_{\mathcal{B}} n} \otimes_{\mathcal{B}} S^{\otimes_{\mathcal{B}} n} \right)$$
$$= \kappa_2^{\mathcal{D}} \left(S^* \otimes_{\mathcal{B}} E_{\mathcal{D}}((S^*)^{n-1} S^{n-1}) S \right)$$
$$= (aL^* + bL)(E_{\mathcal{D}}((S^*)^{n-1} S^{n-1})) = (aL^* + bL)(F_{n-1}),$$

so we get the following recursive algorithm for determining the F_n 's.

$$\begin{cases} F_0(x) = 1\\ F_n(x) = aL^*(F_{n-1})(x) + bL(F_{n-1})(x), & x \in [0,1] \end{cases},$$

where $L^*(f): x \mapsto \int_0^x f(y) dy$ and $L(f): x \mapsto \int_x^1 f(y) dy$. Observe that

$$\frac{\mathrm{d}}{\mathrm{d}x}L(f)(x) = -f(x)$$
 and $\frac{\mathrm{d}}{\mathrm{d}x}L^*(f)(x) = f(x),$

and that

$$F_n(0) = aL^*(F_{n-1})(0) + bL(F_{n-1})(0) = b\int_0^1 F_{n-1}(x)dx = b\tau(F_{n-1})$$

for $n \ge 1$. Using (4.22) we have the following differential equation and initial condition in x

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}F(t,x) = (a-b)tF(t,x), & x \in [0,1] \\ F(t,0) = f(t), \end{cases}$$

where the function f is given by

$$f(t) = F(t,0) = \sum_{n=0}^{\infty} F_n(0)t^n$$

= $1 + \sum_{n=1}^{\infty} \left(aL^*(F_{n-1})(0) + bL(F_{n-1})(0) \right)t^n$
= $1 + b\sum_{n=1}^{\infty} \left(\int_0^1 F_{n-1}(x) dx \right)t^n$
= $1 + bt \int_0^1 \left(\sum_{n=1}^{\infty} F_{n-1}(x)t^{n-1} \right) dx$
= $1 + bt\tau(F(t, \cdot))$

We thus have the unique solution

(4.23)
$$F(t,x) = f(t)e^{(a-b)tx},$$

where we can now use (4.23) and the initial condition to find the function f.

$$f(t) = 1 + bt \int_0^1 F(t, x) dx$$

= 1 + bt $\left[\frac{f(t)}{(a-b)t} e^{(a-b)tx} \right]_0^1 = 1 + bf(t) \frac{\left(e^{(a-b)t} - 1 \right)}{a-b}.$

Hence

$$f(t) = \frac{a-b}{a-be^{(a-b)t}}$$

so that

$$F(t,x) = \frac{(a-b)\mathrm{e}^{(a-b)tx}}{a-\mathrm{b}\mathrm{e}^{(a-b)t}}.$$

Now observe that

$$\begin{split} \left\|\sum_{n=0}^{\infty} \lambda^n S^n\right\|_2^2 &= \tau \left(F(|\lambda|^2, x)\right) \\ &= \int_0^1 F(|\lambda|^2, x) \mathrm{d}x = \frac{1}{|\lambda|^2} \frac{\mathrm{e}^{(a-b)|\lambda|^2} - 1}{a - b\mathrm{e}^{(a-b)|\lambda|^2}} \end{split}$$

Theorem 4.7. The operator $T + \sqrt{\epsilon}Y$ is not a DT-operator.

Proof. By substituting $a = 1 + \epsilon$ and $b = \epsilon$ in lemma 4.6 we have

(4.24)
$$\left\|\sum_{n=0}^{\infty}\lambda^{n}(T+\sqrt{\epsilon}Y)^{n}\right\|_{2}^{2} = \frac{1}{|\lambda|^{2}}\frac{\mathrm{e}^{|\lambda|^{2}}-1}{1+\epsilon-\epsilon\mathrm{e}^{|\lambda|^{2}}}$$

for all λ in a neighborhood of 0. If $T + \sqrt{\epsilon}Y$ is a DT-operator, then by Theorem 4.3 and (4.21), there exists a c > 0, such that when $d = \log(1 + \frac{1}{\epsilon})^{-\frac{1}{2}}$

(4.25)
$$\left\|\sum_{n=0}^{\infty} \lambda^n (T + \sqrt{\epsilon}Y)^n\right\|_2^2 = \frac{1}{c^2 |\lambda|^2} \left((1 - d^2 |\lambda|^2)^{-\frac{c^2}{d^2}} - 1 \right)$$

for all λ in a neighborhood of 0. Consider the two analytic functions,

$$f(s) = \frac{e^s - 1}{1 + \epsilon - \epsilon e^s},$$

$$g(s) = \frac{1}{c^2} \left(1 - d^2 s \right)^{-\frac{c^2}{d^2}} - 1 \right)$$

which are both defined in the complex disc $U = B(0, \log(1 + \frac{1}{\epsilon})^{-\frac{1}{2}})$. By (4.24) and (4.25) f(s) = g(s) for s in some real interval of the form $(0, \delta)$ and hence f(s) = g(s) for all $s \in U$. Moreover f has a meromorphic extension to the full complex plane with a simple pole at $s_0 = \log(1 + \frac{1}{\epsilon})$. Hence g also has a meromorphic extension to the full complex plane with a simple pole at $\log(1 + \frac{1}{\epsilon}) = d^{-2}$. This implies c = d. In this case

$$g(s) = \frac{1}{d^2} \left((1 - d^2 s)^{-1} - 1 \right)$$

which is analytic in $\mathbb{C} \setminus \{s_0\}$. However f has infinitely many poles, namely

$$s_p = \log\left(1 + \frac{1}{\epsilon}\right) + p2\pi, \quad p \in \mathbb{Z}.$$

Since the meromorphic extensions of f and g must coincide, we have reached a contradiction. Therefore $T + \sqrt{\epsilon}Y$ is not a DT-operator.

5. ŚNIADY'S MOMENT FORMULAS. THE CASE k = 2.

Let $k \in \mathbb{N}$ be fixed, and let $(P_{k,n})_{n=0}^{\infty}$ be the sequence of polynomials defined recursively by

(5.1)
$$\begin{cases} P_{k,n}(x) = 1, \\ P_{k,n}^{(k)}(x) = P_{k,n-1}(x+1), & n = 1, 2, \dots, \\ P_{k,n}(0) = P_{k,n}^{(1)}(0) = \cdots P_{k,n}^{(k-1)}(0) = 0, & n = 1, 2, \dots \end{cases}$$

where $P_{k,n}^{(l)}$ denotes the *l*'th derivative of $P_{k,n}$. As in the previous sections, *T* denotes the quasinilpotent *DT* operator. Siniady's main results from [9] are:

Theorem 5.1. [9, Theorem 5 and Theorem 7] (a) For all $k, n \in \mathbb{N}$:

(a) FOT all
$$\kappa, n \in \mathbb{N}$$
.

(5.2)
$$E_{\mathcal{D}}\left(\left((T^*)^k T^k\right)^n\right)(x) = P_{k,n}(x), \quad x \in [0,1].$$

(b) For all $k, n \in \mathbb{N}$:

(5.3)
$$\tau\left(((T^*)^k T^k)^n\right) = \frac{n^{nk}}{(nk+1)!}$$

Actually Sniady considers $E_{\mathcal{D}}((T^k(T^*)^k)^n)$ instead of $E_{\mathcal{D}}(((T^*)^kT^k)^n)$, but it is easily seen, that Theorem 5.1 (a) is equivalent to [9, Theorem 5], by the simple change of variable $x \mapsto 1 - x$.

Sniady's proof of Theorem 5.1 is a very technical combinatorial proof. In this and the following section we will give an analytical proof of Theorem 5.1 based on Voiculescu's \mathcal{R} -transform with amalgamation.

As in [5, (2.11)] we put

$$\rho(z) = -W_0(-z), \qquad z \in \mathbb{C} \setminus [\frac{1}{e}, \infty),$$

where W_0 is the principal branch of Lambert's W-function. Then ρ is the principal branch of the inverse function of $z \mapsto ze^{-z}$. We shall need the following result from [5, Prop. 4.2].

Lemma 5.2. [5, Prop. 4.2] Let $(P_{k,n})_{n=0}^{\infty}$ be a sequence of polynomials given by (5.1). Put for $s \in \mathbb{C}$, $|s| < \frac{1}{e}$ and $j = 1, \ldots, k$

(5.4)
$$\alpha_j(s) = \rho\left(s \mathrm{e}^{\mathrm{i}\frac{2\pi j}{k}}\right),$$

(5.5)
$$\gamma_j(s) = \begin{cases} \prod_{l \neq j} \frac{\alpha_l(s)}{\alpha_l(s) - \alpha_j(s)}, & 0 < |s| < \frac{1}{e} \\ \frac{1}{k}, & s = 0. \end{cases}$$

Then

(5.6)
$$\sum_{n=0}^{\infty} (ks)^{nk} P_{k,n}(x) = \sum_{j=1}^{k} \gamma_j(s) \mathrm{e}^{k\alpha_j(s)x}$$

for all $x \in \mathbb{R}$ and all $s \in B(0, \frac{1}{e})$.

The case k = 1 of theorem 5.1 is the special case $\lambda = 0$ of theorem 3.2. To illustrate our method of proof of theorem 5.1 for $k \ge 2$, we first consider the case k = 2.

Define $\tilde{T} \in M_4(\mathcal{A})$ by

$$\tilde{T} = \begin{pmatrix} 0 & 0 & 0 & T^* \\ T & 0 & 0 & 0 \\ 0 & T & 0 & 0 \\ 0 & 0 & T^* & 0 \end{pmatrix}.$$

Then $\|\tilde{T}\| = \|T\| = \sqrt{e}$. (cf. [4, Corollary 8.11]) For $\mu \in \mathbb{C}$, $|\mu| < \frac{1}{e}$ we let $z = z(\mu)$, denote the Cauchy transform of \tilde{T} at $\tilde{\mu} = \mu \mathbf{1}_{M_4(\mathcal{A})}$ wrt. amalgamation over $M_4(\mathcal{D})$ i.e.

$$z = E_{\mathcal{D}}\left((\tilde{\mu} - \tilde{T})^{-1} \right).$$

Clearly

(5.7)
$$(\tilde{\mu} - \tilde{T})^{-1} = \sum_{n=0}^{\infty} \mu^{-n-1} \tilde{T}^n = \left(\sum_{n=0}^{3} \mu^{-n-1} \tilde{T}^n\right) \left(\sum_{n=0}^{\infty} \mu^{-4n} \tilde{T}^{4n}\right).$$

By direct computation

$$\tilde{T}^{2} = \begin{pmatrix} 0 & 0 & (T^{*})^{2} & 0 \\ 0 & 0 & 0 & TT^{*} \\ T^{2} & 0 & 0 & 0 \\ 0 & T^{*}T & 0 & 0 \end{pmatrix},$$
$$\tilde{T}^{3} = \begin{pmatrix} 0 & (T^{*})^{2}T & 0 & 0 \\ 0 & 0 & T(T^{*})^{2} & 0 \\ 0 & 0 & 0 & T^{2}T^{*} \\ T^{*}T^{2} & 0 & 0 & 0 \end{pmatrix}$$

and

$$\tilde{T}^4 = \begin{pmatrix} (T^*)^2 T^2 & 0 & 0 & 0\\ 0 & T(T^*)^2 T & 0 & 0\\ 0 & 0 & T^2(T^*)^2 & 0\\ 0 & 0 & 0 & T^* T^2 T^* \end{pmatrix}.$$

Hence using the fact that the expectation $E_{\mathcal{D}}$ of a monomial in T and T^* vanishes unless T and T^* occur the same number of times, we get from (5.7) that z is of the form

(5.8)
$$z = \begin{pmatrix} z_{11} & 0 & 0 & 0\\ 0 & z_{22} & 0 & z_{24}\\ 0 & 0 & z_{33} & 0\\ 0 & z_{42} & 0 & z_{44} \end{pmatrix}$$

where $z_{11}, z_{22}, z_{24}, z_{33}, z_{42}, z_{44} \in \mathcal{D}$ are given by

$$z_{11} = \mu^{-1} E_{\mathcal{D}} ((1 - \mu^{-4} (T^*)^2 T^2)^{-1}),$$

$$z_{22} = \mu^{-1} E_{\mathcal{D}} ((1 - \mu^{-4} T (T^*)^2 T)^{-1}),$$

$$z_{33} = \mu^{-1} E_{\mathcal{D}} ((1 - \mu^{-4} T^2 (T^*)^2)^{-1}),$$

$$z_{44} = \mu^{-1} E_{\mathcal{D}} ((1 - \mu^{-4} T^* T^2 T^*)^{-1}),$$

$$z_{24} = \mu^{-3} E_{\mathcal{D}} (T (1 - \mu^{-4} (T^*)^2 T^2)^{-1} T^*),$$

$$z_{42} = \mu^{-3} E_{\mathcal{D}} (T^* (1 - \mu^{-4} T^2 (T^*)^2)^{-1} T).$$

For the last 2 identities, we have used, that

$$A(1 - \eta BA)^{-1} = (1 - \eta AB)^{-1}A$$

for $A, B \in \mathcal{A}$ and $\eta \in \mathbb{C}$ whenever both sides of this equality are welldefined.

By lemma 2.1, we know, that there exists a $\delta > 0$ such that when $w \in M_4(\mathcal{D})_{\text{inv}}$ and $\mu \in \mathbb{C}$ satisfies $||w|| < \delta$, $|\mu| > \frac{1}{\delta}$ and

(5.9)
$$\mathcal{R}^{M_4(\mathcal{D})}_{\tilde{T}}(w) + w^{-1} = \mu \mathbf{1}_{M_4(\mathcal{A})}$$

then $w = E_{M_4(\mathcal{D})}((\tilde{\mu} - \tilde{T})^{-1}) = z$. In particular

$$w_{11} = z_{11} = \mu^{-1}((1 - \mu^{-4}(T^*)^2 T^2)^{-1}),$$

Hence, if we can find a suitable solution to (5.8) for all $\mu \in \mathbb{C}$ in a neighborhood of ∞ , we can find $E_{\mathcal{D}}(((T^*)^2T^2)^n)$ for $n = 1, 2, \ldots$ by determining the power series expansion of w_{11} as a function of μ^{-1} .

Since (T, T^*) is a \mathcal{D} -Gaussian pair by [5, Appendix] it follows from lemma 2.2 that

$$\kappa_n^{M_4(\mathcal{D})}((m_1 \otimes a_1) \otimes_{M_4(\mathcal{D})} \cdots \otimes_{M_4(\mathcal{D})} (m_n \otimes a_n)) = 0$$

when $n \neq 2, m_1, m_2, ..., m_n \in M_4(\mathbb{C})$ and $a_1, a_2, ..., a_n \in \{T, T^*\}$. By definition

$$\tilde{T} = (e_{21} + e_{32}) \otimes T + (e_{43} + e_{14}) \otimes T$$

so by linearity of $\kappa_n^{M_4(\mathcal{D})}$, it follows that

$$\kappa_n^{M_4(\mathcal{D})}(\tilde{T}\otimes_{M_4(\mathcal{D})}\cdots\otimes_{M_4(\mathcal{D})}\tilde{T})=0$$

when $n \neq 2$ i.e. \tilde{T} is $M_4(\mathcal{D})$ -Gaussian.

Hence using (2.4) we get

$$\begin{aligned} \mathfrak{R}_{\tilde{T}}^{M_4(\mathcal{D})}(w) &= \kappa_2^{M_4(\mathcal{D})}(\tilde{T} \otimes_{M_4(\mathcal{D})} w\tilde{T}) = E_{M_4(\mathcal{D})} \left(\tilde{T} w\tilde{T} \right) \\ &= E_{M_4(\mathcal{D})} \left(\begin{pmatrix} T^* w_{42} T & 0 & T^* w_{44} T^* & 0 \\ 0 & 0 & 0 & T w_{11} T^* \\ T w_{22} T & 0 & T w_{24} T^* & 0 \\ 0 & T^* w_{33} T & 0 & 0 \end{pmatrix} \right) \end{aligned}$$

for $w = (w_{ij})_{i,j=1,...,4} \in M_4(\mathcal{D}).$

Since $E_{\mathcal{D}}(TfT) = E_{\mathcal{D}}(T^*fT^*) = 0$, and $E_{\mathcal{D}}(T^*fT) = L^*(f)$, $E_{\mathcal{D}}(TfT^*) = L(f)$ for $f \in L^{\infty}([0,1])$, we have:

$$\mathcal{R}_{\tilde{T}}^{M_4(\mathcal{D})}(w) = \begin{pmatrix} L^*(w_{42}) & 0 & 0 & 0\\ 0 & 0 & 0 & L(w_{11})\\ 0 & 0 & L(w_{24}) & 0\\ 0 & L^*(w_{33}) & 0 & 0 \end{pmatrix}$$

for $w \in M_4(\mathcal{D})$. By (5.8) we only have to consider w of the form

(5.10)
$$w = \begin{pmatrix} w_{11} & 0 & 0 & 0\\ 0 & w_{22} & 0 & w_{24}\\ 0 & 0 & w_{33} & 0\\ 0 & w_{42} & 0 & w_{44} \end{pmatrix}.$$

For $w \in M_4(\mathcal{D})_{inv}$ of the form (5.10), (5.9) reduces to the three equations

(5.11)
$$\begin{cases} L^*(w_{42}) + \frac{1}{w_{11}} = \mu 1_{\mathcal{D}} \\ \begin{pmatrix} 0 & L(w_{11}) \\ L^*(w_{33}) & 0 \end{pmatrix} + \begin{pmatrix} w_{22} & w_{24} \\ w_{42} & w_{44} \end{pmatrix}^{-1} = \mu 1_{M_2(\mathcal{D})} \\ L(w_{24}) + \frac{1}{w_{33}} = \mu 1_{\mathcal{D}} \end{cases}$$

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Definition 5.3. Let $f \in C([0,1])$. We call $(f^{(-n)})_{n=1}^l$ for the succesive antiderivatives of f if

$$\frac{\mathrm{d}}{\mathrm{d}x}(f^{(-n)}) = f^{(1-n)} \text{ for } n = 2, 3, \dots, l$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}(f^{(-1)}) = f.$$

Lemma 5.4. Let $f \in C^2([0,1])$ and let $f^{(-1)}$ and $f^{(-2)}$ be the successive antiderivatives of f for which

(i)
$$f^{(-1)}(1) = 0$$
, $f^{(-2)}(1) = \mu^3$.
Assume further, that
(ii) $f(0) = \mu^{-1}$ and $f^{(1)}(0) = 0$.
(iii) For all $x \in [0, 1]$,

$$\begin{aligned} f(x) \neq 0 \\ \begin{vmatrix} f^{(-1)}(x) & f(x) \\ f(x) & f^{(1)}(x) \end{vmatrix} \neq 0 \end{aligned}$$

while

$$\begin{vmatrix} f^{(-2)}(x) & f^{(-1)}(x) & f(x) \\ f^{(-1)}(x) & f(x) & f^{(1)}(x) \\ f(x) & f^{(1)}(x) & f^{(2)}(x) \end{vmatrix} = 0.$$

Then $w_{11}, w_{22}, w_{33}, w_{44}, w_{24}, w_{42} \in C([0, 1])$ given by

(5.12)
$$\begin{cases} w_{11} = f \\ w_{22} = w_{44} = -\frac{1}{\mu} \frac{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}}{f^2} \\ w_{24} = \frac{1}{\mu^2} \frac{f^{(-1)} \begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}}{f^2} \\ w_{42} = \frac{f^{(1)}}{f^2} \\ w_{33} = \mu^2 \frac{f \begin{vmatrix} f & f^{(1)} \\ f^{(1)} & f^{(2)} \end{vmatrix}}{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}}^2 \end{cases}$$

is a solution to (5.11). Moreover

(5.13)
$$\begin{vmatrix} w_{22} & w_{24} \\ w_{42} & w_{44} \end{vmatrix} = -\frac{1}{\mu^2} \frac{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}}{f^2}$$

and

$$L^*(w_{42}) = \mu - \frac{1}{f}
 L^*(w_{33}) = -\mu^2 \frac{f^{(1)}}{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}}$$

Proof. Assume $w_{11}, w_{22}, w_{33}, w_{44}, w_{24}, w_{42}$ is given by (5.12). Then (5.13) follows immediately. Note that for $f \in C([0, 1])$, the functions g = L(f) and $h = L^*(f)$ are characterized by

$$g^{(1)} = -f$$
 and $g(1) = 0$
 $h^{(1)} = f$ and $h(0) = 0.$

Hence (5.14) is equivalent to (5.15) and (5.16) below.

(5.15)
$$\begin{cases} \frac{d}{dx}f^{(-1)} = w_{11} \\ \frac{d}{dx}\left(\frac{1}{\mu^2} \frac{\left| f^{(-2)} f^{(-1)} \right|}{f^{(-1)} f} \right| \\ \frac{d}{dx}\left(-\frac{1}{\mu^2} \frac{f^{(-1)}}{f} \right) = w_{24} \\ \frac{d}{dx}\left(-\frac{1}{f}\right) = w_{42} \\ \frac{d}{dx}\left(-\mu^2 \frac{f^{(1)}}{\left| f^{(-1)} f \right|} \right) = w_{33} \\ \frac{d}{dx}\left(-\mu^2 \frac{f^{(1)}}{\left| f^{(-1)} f \right|} \right) = w_{33} \\ \begin{cases} f^{(-1)}(1) = 0, & \frac{\left| f^{(-2)}(1) f^{(-1)}(1) \right|}{f(1)} \\ \frac{1}{f(0)} = \mu, & f^{(1)}(0) = 0 \end{cases}$$

Now, (5.16) is trivial from (i) and (ii). Next we prove (5.15): Clearly

$$\frac{\mathrm{d}}{\mathrm{d}x}f^{(-1)} = f = w_{11}$$
 and $\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{1}{f}\right) = \frac{f^{(1)}}{f^2} = w_{42}.$

Moreover

(5.17)
$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\begin{vmatrix} f^{(-2)} & f^{(-1)} \\ f^{(-1)} & f \end{vmatrix}}{f} \right) \\ = \frac{f \begin{vmatrix} f^{(-2)} & f \\ f^{(-1)} & f^{(1)} \end{vmatrix} - f^{(1)} \begin{vmatrix} f^{(-2)} & f^{(-1)} \\ f^{(-1)} & f \end{vmatrix}}{f^2} = \frac{f^{(-1)} \begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}}{f^2} = \mu^2 w_{24}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{f^{(1)}}{\left| f^{(-1)} \quad f \right|} \right) = \frac{\left| \begin{array}{c} f^{(-1)} \quad f \\ f \quad f^{(1)} \end{array} \right| f^{(2)} - \left| \begin{array}{c} f^{(-1)} \quad f \\ f^{(1)} \quad f^{(2)} \end{array} \right|}{\left| \begin{array}{c} f^{(-1)} \quad f \\ f \quad f^{(1)} \end{array} \right|^2} \\ = -\frac{f \left| \begin{array}{c} f \quad f^{(1)} \\ f^{(1)} \quad f^{(2)} \\ \end{array} \right|}{\left| \begin{array}{c} f^{(-1)} \quad f \\ f^{(2)} \end{array} \right|^2} = -\frac{1}{\mu^2} w_{33}.$$

Hence (5.15) holds. It remains to be proved that $w_{11}, w_{22}, w_{33}, w_{44}, w_{24}, w_{42}$ is a solution to (5.11). By (5.12) and (5.14), we have

$$L^*(w_{42}) + \frac{1}{w_{11}} = \left(\mu - \frac{1}{f}\right) + \frac{1}{f} = \mu.$$

Moreover by (5.12) and (5.13)

$$\begin{pmatrix} w_{22} & w_{24} \\ w_{42} & w_{44} \end{pmatrix}^{-1} = \frac{1}{w_{22}w_{44} - w_{24}w_{42}} \begin{pmatrix} w_{44} & -w_{24} \\ -w_{42} & w_{22} \end{pmatrix}$$
$$= \begin{pmatrix} \mu & f^{(-1)} \\ \mu^2 \frac{f^{(1)}}{\left| f^{(-1)} & f \right|} & \mu \\ f & f^{(1)} \end{vmatrix}$$

which proves that the first and the second inequality in (5.11).

By (5.12) and (5.14),

$$w_{33}(\mu - L(w_{24})) = \frac{\begin{vmatrix} f & f^{(1)} \\ f^{(1)} & f^{(2)} \end{vmatrix} \begin{vmatrix} f^{(-2)} & f^{(-1)} \\ f^{(-1)} & f \end{vmatrix}}{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}^2} = 1 + \frac{\sigma}{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}^2}$$

where

$$\sigma = \begin{vmatrix} f & f^{(1)} \\ f^{(1)} & f^{(2)} \end{vmatrix} \begin{vmatrix} f^{(-2)} & f^{(-1)} \\ f^{(-1)} & f \end{vmatrix} - \begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}^2 = f \begin{vmatrix} f^{(-2)} & f^{(-1)} & f \\ f^{(-1)} & f & f^{(1)} \\ f & f^{(1)} & f^{(2)} \end{vmatrix}.$$

Hence by (iii), $\sigma = 0$. Therefore $w_{33}(x) \neq 0$ for all $x \in [0, 1]$ and $w_{33}^{-1} = \mu - L(w_{24})$, proving the last equality in (5.11).

Lemma 5.5. Let $\alpha_j(s), \gamma_j(s)$ for j = 1, 2 be as in lemma 5.2 for k = 2, i.e. $\alpha_1(0) = \alpha_2(0) = 0, \ \gamma_1(0) = \gamma_2(0) = \frac{1}{2}$ and for $0 < |s| < e^{-1}$:

$$\begin{aligned}
\alpha_1(s) &= \rho(s), & \alpha_2(s) = \rho(-s), \\
\gamma_1(s) &= \frac{\alpha_1(s)}{\alpha_1(s) - \alpha_2(s)}, & \gamma_2(s) = \frac{\alpha_2(s)}{\alpha_2(s) - \alpha_1(s)}
\end{aligned}$$

Let $\mu \in \mathbb{C}$, $|\mu| > \sqrt{e}$, put $s = \frac{1}{2}\mu^{-2}$ and

(5.18)
$$f(x) = \frac{1}{\mu} \left(\sum_{j=1}^{2} \gamma_j(s) e^{2\alpha_j(s)x} \right), \quad x \in \mathbb{R}$$

(5.19)
$$f^{(-1)}(x) = \frac{1}{2\mu} \left(\sum_{j=1}^{2} \frac{\gamma_j(s)}{\alpha_j(s)} e^{2\alpha_j(s)x} \right), \quad x \in \mathbb{R}$$

(5.20)
$$f^{(-2)}(x) = \frac{1}{4\mu} \left(\sum_{j=1}^{2} \frac{\gamma_j(s)}{\alpha_j(s)^2} e^{2\alpha_j(s)x} \right), \quad x \in \mathbb{R}$$

Then

(i) $f^{(-1)}$, $f^{(-2)}$ are successively antiderivatives of f, (5.21) $f^{(-1)}(1) = 0$, $f^{(-2)}(1) = \mu^3$

and

(5.22)
$$f(0) = \mu^{-1}, \quad f^{(1)}(0) = 0.$$

(ii) The following asymptotic formulas holds for $|\mu| \to \infty$:

$$f^{(-2)}(x) = \mu^{3} + \mathcal{O}(\mu^{-1})$$

$$f^{(-1)}(x) = (x - 1)\mu^{-1} + \mathcal{O}(\mu^{-5})$$

$$f(x) = \mu^{-1} + \mathcal{O}(\mu^{-5})$$

$$f^{(1)}(x) = x\mu^{-5} + \mathcal{O}(\mu^{-9})$$

$$f^{(2)}(x) = x\mu^{-5} + \mathcal{O}(\mu^{-9})$$

where the error estimates holds uniformly in x on a compact subset in \mathbb{R} . There exists $\mu_{2} > \sqrt{2}$ such that the restriction of f to [0, 1] satisfies all

(iii) There exists $\mu_0 \ge \sqrt{e}$ such that the restriction of f to [0,1] satisfies all the conditions in lemma 5.4, when $|\mu| > \mu_0$.

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Proof. Clearly $f^{(-1)}$ and $f^{(-2)}$ are successively antiderivatives of f and

$$f(0) = \frac{1}{\mu} \sum_{j=1}^{2} \gamma_j(s) = \frac{1}{\mu}$$
$$f^{(1)}(0) = \frac{2}{\mu} \sum_{j=1}^{2} \alpha_j(s) \gamma_j(s) = 0.$$

To prove (5.21), note first, that since $\rho : \mathbb{C} \setminus [\frac{1}{e}, \infty) \to \mathbb{C}$ is a branch of the inverse function of $z \mapsto z e^{-z}$, we have

$$\rho(w)e^{-\rho(w)} = w, \ |w| < \frac{1}{e}$$

and therefore

$$e^{2\alpha_j(s)} = \frac{\alpha_j(s)^2}{s^2}, \ j = 1, 2.$$

Since $s^2 = \frac{1}{4}\mu^{-4}$, it follows that

(5.23)
$$f^{(-2)}(x+1) = \mu^4 f(x), \quad x \in \mathbb{R}$$

(5.24)
$$f^{(-1)}(x+1) = \mu^4 f^{(1)}(x), \quad x \in \mathbb{R}$$

(5.24)
$$f^{(-1)}(x+1) = \mu^4 f^{(1)}(x), \quad x \in \mathbb{R}$$

(5.25)
$$f(x+1) = \mu^4 f^{(2)}(x), \quad x \in \mathbb{R}.$$

In particular

$$f^{(-2)}(1) = \mu^4 f(0) = \mu^3$$
$$f^{(-1)}(1) = \mu^4 f^{(1)}(0) = 0.$$

By the proof of [5, Prop. 4.2], $\alpha_j(s)$ and $\rho_j(s)$ are continuous functions of $s \in B(0, \frac{1}{e})$. Hence, regarding f as a function of μ ,

$$\lim_{|\mu| \to \infty} (\mu f(x)) = \sum_{j=1}^{2} \gamma_j(0) e^{2\alpha_j(0)x} = 1$$

where the limit holds uniformly in x on compact subsets of \mathbb{R} . Hence by (5.25) $f^{(2)}(x) = \mathcal{O}(\mu^{-5})$ as $|\mu| \to \infty$ uniformly in x on compact subsets of \mathbb{R} . By (5.22),

(5.26)
$$f^{(1)}(x) = \int_0^x f^{(2)}(t) dt$$

(5.27)
$$f(x) = \mu^{-1} + \int_0^x f^{(1)}(t) dt$$

which implies, that $f^{(1)}(x) = O(\mu^{-5})$ and (7.20) $f(x) = \mu^{-1} + O(\mu^{-5})$

(5.28)
$$f(x) = \mu^{-1} + \mathcal{O}(\mu^{-3})$$

uniformly in x on compact subsets of \mathbb{R} .

Using again (5.25), (5.26) and (5.27), we get

$$f^{(2)}(x) = \mu^{-5} + \mathcal{O}(\mu^{-9})$$
$$f^{(1)}(x) = x\mu^{-5} + \mathcal{O}(\mu^{-9}).$$

By (5.21)

$$f^{(-1)}(x) = \int_{1}^{x} f(t) dt$$
$$f^{(-2)}(x) = \mu^{3} + \int_{1}^{x} f^{(-1)}(t) dt$$

Hence by (5.28),

$$f^{(-1)}(x) = (x-1)\mu^{-1} + \mathcal{O}(\mu^{-5})$$
$$f^{(-2)}(x) = \mu^3 + \mathcal{O}(\mu^{-1})$$

where all estimates holds uniformly on compact subsets of \mathbb{R} . This proves (ii). By (i), $f^{(-1)}$, $f^{(-2)}$ coinside with the succesive antiderivatives of f considered in lemma 5.4 and $f(0) = \mu^{-1}$, $f^{(1)}(0) = 0$.

Moreover, by (ii),

$$f(x) = \mu^{-1} + \mathcal{O}(\mu^{-5})$$
$$\begin{vmatrix} f^{(-1)}(x) & f(x) \\ f(x) & f^{(1)}(x) \end{vmatrix} = \mu^{-2} + \mathcal{O}(\mu^{-6})$$

where the error terms holds uniformly in $x \in [0, 1]$. Hence there exists $\mu_0 \ge \sqrt{e}$, such that

$$f(x) \neq 0$$
 and $\begin{vmatrix} f^{(-1)}(x) & f(x) \\ f(x) & f^{(1)}(x) \end{vmatrix} \neq 0$

for all $x \in [0, 1]$. Moreover by the matrix factorization

(5.29)
$$\begin{pmatrix} f^{(-2)}(x) & f^{(-1)}(x) & f(x) \\ f^{(-1)}(x) & f(x) & f^{(1)}(x) \\ f(x) & f^{(1)}(x) & f^{(2)}(x) \end{pmatrix} \\ = \begin{pmatrix} 1 & 1 \\ 2\alpha_1(s) & 2\alpha_2(s) \\ 4\alpha_1(s)^2 & 4\alpha_2(s)^2 \end{pmatrix} \begin{pmatrix} \frac{\gamma_1(s)}{4\alpha_1(s)^2} e^{2\alpha_1(s)x} & 0 \\ 0 & \frac{\gamma_2(s)}{4\alpha_2(s)^2} e^{2\alpha_2(s)x} \end{pmatrix} \begin{pmatrix} 1 & 2\alpha_1(s) & 4\alpha_1(s)^2 \\ 1 & 2\alpha_2(s) & 4\alpha_2(s)^2 \end{pmatrix}$$

it follows, that the matrix on the left hand side has rank less than or equal to 2, i.e.

$$\begin{vmatrix} f^{(-2)}(x) & f^{(-1)}(x) & f(x) \\ f^{(-1)}(x) & f(x) & f^{(1)}(x) \\ f(x) & f^{(1)}(x) & f^{(2)}(x) \end{vmatrix} = 0$$

for $x \in [0, 1]$. Hence f satisfies all the conditions in lemma 5.4, when $|\mu| > \mu_0$.

Proof of Theorem 5.1 in the case k = 2: By lemma 2.1 there exists a $\delta > 0$, such that when $w \in M_4(\mathcal{D})_{inv}$ and $\mu \in \mathbb{C}$ satisfies $||w|| < \delta, |\mu| > \frac{1}{\delta}$ and

(5.30)
$$\Re_{\tilde{T}}^{M_4(\mathcal{D})}(w) + w^{-1} = \mu \mathbb{1}_{M_4(\mathcal{D})}$$

then $w = E_{\mathcal{D}}((\tilde{\mu} - \tilde{T})^{-1})$. In particular

(5.31)
$$w_{11} = \mu^{-1} E_{\mathcal{D}} ((1 - \mu^{-4} (T^*)^2 T^2)^{-1}).$$

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Let $\mu \in \mathbb{C}, |\mu| > \sqrt{e}$, put $s = \frac{1}{2}\mu^{-2}$ and

$$f(x) = \frac{1}{\mu} \left(\sum_{j=1}^{2} \gamma_j(s) e^{2\alpha_j(s)x} \right)$$

for $x \in [0, 1]$ as in lemma 5.5. By lemma 5.5 (iii) there exists a $\mu_0 > \sqrt{e}$, such that when $|\mu| > \mu_0$, then f satisfies all the requirements af lemma 5.4. Hence by lemma 5.4, the matrix $w \in M_4(\mathcal{D})$ given by (5.10) and (5.12) is a solution to (5.30). Moreover by the asymptotic formulas in lemma 5.5 (ii),

$$\begin{vmatrix} f^{(-2)}(x) & f^{(-1)}(x) \\ f^{(-1)}(x) & f(x) \end{vmatrix} = \mu^2 + \mathcal{O}(\mu^{-2}),$$
$$\begin{vmatrix} f^{(-1)}(x) & f(x) \\ f(x) & f'(x) \end{vmatrix} = -\mu^{-2} + \mathcal{O}(\mu^{-6}),$$
$$\begin{vmatrix} f(x) & f'(x) \\ f'(x) & f''(x) \end{vmatrix} = \mu^{-6} + \mathcal{O}(\mu^{-10}).$$

Hence by (5.12) and the asymptotic formulas for $f^{(-1)}$, f and f', we have

$$w_{11} = \mu^{-1} + \mathcal{O}(\mu^{-5}),$$

$$w_{22} = w_{44} = \mu^{-1} + \mathcal{O}(\mu^{-5}),$$

$$w_{24} = (1 - x)\mu^{-3} + \mathcal{O}(\mu^{-3}),$$

$$w_{42} = x\mu^{-3} + \mathcal{O}(\mu^{-3}),$$

$$w_{33} = \mu^{-1} + \mathcal{O}(\mu^{-5}),$$

where all the error estimates holds uniformly in $x \in [0, 1]$. Hence, there exists $\mu_1 \ge \max\{\mu_0, \frac{1}{\delta}\}$, such that when $|\mu| > \mu_1$ then $||w|| < \delta$, and hence

$$w = E_{M_4(\mathcal{D})}((\tilde{\mu} - \tilde{T})^{-1}).$$

By (5.12), $w_{11} = f$. Hence by (5.31) and (5.18)

$$E_{\mathcal{D}}((1-\mu^{-4}(T^*)^2T^2)^{-1})(x) = \mu f(x) = \sum_{j=1}^2 \gamma_j(s) e^{2\alpha_j(s)x}$$

where $s = \frac{1}{2}\mu^{-2}$, i.e. for $|s| < \frac{1}{2}\mu_1^{-2}$,

$$E_{\mathcal{D}}((1-(2s)^2(T^*)^2T^2)^{-1})(x) = \sum_{j=1}^2 \gamma_j(s) e^{2\alpha_j(s)x}$$

and therefore

(5.32)
$$\sum_{j=0}^{\infty} (2s)^{2n} E_{\mathcal{D}}(((T^*)^2 T^2)^n)(x) = \sum_{j=1}^2 \gamma_j(s) e^{2\alpha_j(s)x}$$

Hence by lemma 5.2 and by the uniqueness of the power series expansions of analytic functions, we have

$$E_{\mathcal{D}}(((T^*)^2 T^2)^n)(x) = P_{2,n}(x)$$

for $n \in \mathbb{N}$ and $x \in [0, 1]$. This proves theorem 5.1(a) in the case k = 2. Theorem 5.1 (b) also follows from (5.32) by integrating the right hand side of (5.32) from 0 to 1 with respect to x (cf. [5, remark 4.3]).

6. ŚNIADY'S MOMENT FORMULAS. THE GENERAL CASE.

The above proof of Theorem 5.1 in the case k = 2 can fairly easily be generalized to all $k \ge 2$ (Recall that the case k = 1 is contained in theorem 3.2).

Let $k \geq 2$ and define $\tilde{T} \in M_{2k}(\mathcal{A})$ by

$$\tilde{T} = \sum_{j=1}^{k} (T \otimes e_{j+1,j} + T^* \otimes e_{k+j+1,k+j})$$

where the indices are computed modulo 2k, such that $e_{2k+1,2k} = e_{1,2k}$. For $\mu \in \mathbb{C}, |\mu| < \frac{1}{\sqrt{e}}$, we put $\tilde{\mu} = \mu \mathbf{1}_{2k}$ and

$$z = z(\mu) = E_{M_{2k}(\mathcal{D})}((\tilde{\mu} - \tilde{T})^{-1})$$

Then only the diagonal entries $z_{11}, \ldots, z_{2k,2k}$ and the off-diagonal entries $z_{2,2k}$, $z_{3,2k-1}, \ldots, z_{2k,2}$ can be non-zero. Moreover,

$$z_{11} = \mu^{-1} E_{\mathcal{D}}((1 - \mu^{-2k} (T^*)^k T^k)^{-1}).$$

The operator \tilde{T} is $M_{2k}(\mathcal{D})$ -Gaussian, and repeating the arguments for k = 2, we get that for $w \in M_{2k}(\mathcal{D})$, the matrix

(6.1)
$$u = \mathcal{R}_{\tilde{T}}^{M_{2k}(\mathcal{D})}(w)$$

can have at most 2k non-zero entries, namely the entries

$$u_{11} = L^{*}(w_{2k,2})$$

$$u_{2k,2} = L^{*}(w_{2k-1,3})$$

$$\vdots \qquad \vdots$$

$$u_{k+2,k} = L^{*}(w_{k+1,k+1})$$

$$u_{k+1,k+1} = L(w_{k,k+2})$$

$$u_{k,k+2} = L(w_{k-1,k+3})$$

$$\vdots \qquad \vdots$$

$$u_{2,2k} = L(w_{1,1}).$$

By lemma 2.1 there exists a $\delta > 0$ (depending on k), such that if $w \in M_{2k}(\mathcal{D})_{inv}, ||w|| < \delta, \mu \in \mathbb{C}, |\mu| > \frac{1}{\delta}$ and

(6.3)
$$\mathfrak{R}^{M_{2k}(\mathcal{D})}_{\tilde{T}}(w) + w^{-1} = \mu \mathbf{1}_{M_{2k}(\mathcal{D})},$$

then

$$w = z = E_{M_{2k}(\mathcal{D})}((\tilde{\mu} - T)^{-1}).$$

In particular

$$w_{11} = \mu^{-1} E_{\mathcal{D}}((1 - \mu^{-2k} (T^*)^k T^k)^{-1}).$$

Next we construct an explicit solution to (6.3). By the above remarks on z, it is sufficient to consider those $w \in M_{2k}(\mathcal{D})_{inv}$ for which only the entries $z_{11}, \ldots, z_{2k,2k}$ and $z_{2,2k}, z_{3,2k-1}, \ldots, z_{2k,2}$ can be non-zero. For such w, (6.3) can by (6.1) and (6.2) be reduced to the k + 1 identities:

(6.4)
$$\begin{cases} L^*(w_{2k,2}) + \frac{1}{w_{11}} = \mu 1_{\mathcal{D}} \\ \begin{pmatrix} 0 \\ L^*(w_{2k-1-j,j+3}) & 0 \end{pmatrix} + \begin{pmatrix} w_{2+j,2+j} & w_{2+j,2k-j} \\ w_{2k-j,2+j} & w_{2k-j,2k-j} \end{pmatrix}^{-1} = \mu 1_{M_2(\mathcal{D})}, \\ j = 0, 1, \dots, k-2, \\ L(w_{k,k+2}) + \frac{1}{w_{k+1,k+1}} = \mu 1_{\mathcal{D}}. \end{cases}$$

Definition 6.1. For $j \in \mathbb{N} \cup \{0\}$ and $g \in C^{2j+2}$, we let $\Delta_j(g)$ denote the determinant

(6.5)
$$\Delta_{j}(g) = \begin{vmatrix} g & g^{(1)} & \ddots & g^{(j)} \\ g^{(1)} & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & g^{(2j-1)} \\ g^{(j)} & \ddots & g^{(2j-1)} & g^{(2j)} \end{vmatrix}$$

In particular $\Delta_0(g) = g$.

Lemma 6.2. Let
$$g \in C^{2j+2}(\mathbb{R})$$
 and $j \in \mathbb{N}$. Then
(6.6) $\Delta_j(g^{(2)})\Delta_j(g) - \Delta_j(g^{(1)})^2 = \Delta_{j-1}(g^{(2)})\Delta_{j+1}(g)$

and

(6.7)
$$\Delta_{j-1}(g^{(2)})\frac{\mathrm{d}}{\mathrm{d}x}(\Delta_j(g)) - \Delta_j(g)\frac{\mathrm{d}}{\mathrm{d}x}(\Delta_{j-1}(g^{(2)})) = \Delta_{j-1}(g^{(1)})\Delta_j(g^{(1)}).$$

The proof of lemma 6.2 relies on elementary matrix manipulations and is contained in lemma A.1 of appendix A. More specifically (6.6) is a direct consequence of (a) from lemma A.1, and (6.7) follows from (b) of lemma A.1 by using the elementary fact that:

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\Delta_j(g) \right) = \begin{vmatrix} g & g^{(1)} & \ddots & g^{(j)} \\ g^{(1)} & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & g^{(2j-1)} \\ g^{(j-1)} & \ddots & g^{(2j-2)} & g^{(2j-1)} \\ g^{(j+1)} & \ddots & g^{(2j)} & g^{(2j+1)} \end{vmatrix},$$

that is, differentiating (6.5) is the same as differentiating the last row of (6.5).

The next two lemmas are the generalizations of lemma 5.4 and lemma 5.5 to arbitrary $k \ge 2$.

Lemma 6.3. Let $f \in C^k([0,1])$ and let $(f^{(-j)})_{j=1}^k$ be the antiderivatives of f for which,

(i)

$$f^{(-j)}(1) = \begin{cases} 0, & 1 \le j \le k-1, \\ \mu^{2k-1}, & j = k. \end{cases}$$

(ii) Assume further that

$$f(0) = \mu^{-1}$$
 and $f^{(-j)}(0) = 0$ for $1 \le j \le k - 1$.

(iii) For all $x \in [0, 1]$,

$$\Delta_j(f^{(-j)})(x) \neq 0, \text{ for } j = 0..., k-1$$

and

$$\Delta_k(f^{(-k)})(x) = 0$$

Then the set of 4k - 2 functions listed in (6.8), (6.9) and (6.10) below is a solution to (6.4).

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(6.8)
$$\begin{cases} w_{11} = f \\ w_{22} = w_{2k,2k} = -\frac{1}{\mu} \frac{\Delta_1(f^{(-1)})}{f^2} \\ w_{2,2k} = \frac{1}{\mu^2} \frac{f^{(-1)} \Delta_1(f^{(-1)})}{f^2} \\ w_{2k,2} = \frac{f^{(1)}}{f^2} \end{cases}$$

For j = 1, ..., k - 2

(6.9)
$$\begin{cases} w_{j+2,j+2} = w_{2k-j,2k-j} = -\frac{1}{\mu} \frac{\Delta_{j-1}(f^{(1-j)})\Delta_{j+1}(f^{(-1-j)})}{\Delta_j(f^{(-j)})^2} \\ w_{j+2,2k-j} = \frac{1}{\mu^{2j+2}} \frac{\Delta_j(f^{(-1-j)})\Delta_{j+1}(f^{(-1-j)})}{\Delta_j(f^{(-j)})^2} \\ w_{2k-j,j+2} = \mu^{2j} \frac{\Delta_{j-1}(f^{(1-j)})\Delta_j(f^{(1-j)})}{\Delta_j(f^{(-j)})^2} \end{cases}$$

(6.10)
$$w_{k+1,k+1} = \mu^{2k+2} \frac{\Delta_{k-2}(f^{(2-k)})\Delta_{k-1}(f^{(2-k)})}{\Delta_{k-1}(f^{(1-k)})^2}$$

Moreover for $j = 0, \ldots, k - 2$

(6.11)
$$\begin{vmatrix} w_{j+2,j+2} & w_{j+2,2k-2} \\ w_{2k-j,j+2} & w_{2k-j,2k-j} \end{vmatrix} = \frac{1}{\mu} w_{j+2,j+2}$$

and

(6.12)
$$\begin{cases} L(w_{11}) = -f^{(-1)} \\ L(w_{j+2,2k-j}) = -\frac{1}{\mu^{2j+2}} \frac{\Delta_{j+1}(f^{(-2-j)})}{\Delta_j(f^{(-j)})}, & 0 \le j \le k-3 \\ L(w_{k,k+2}) = \mu - \frac{1}{\mu^{2k-2}} \frac{\Delta_{k-1}(f^{(-k)})}{\Delta_{k-2}(f^{(2-k)})} \\ \\ \\ L^*(w_{2k,2}) = \mu - \frac{1}{f} \\ L^*(w_{2k-j,2+j}) = -\mu^{2j} \frac{\Delta_{j-1}(f^{(-2-j)})}{\Delta_j(f^{(-j)})}, & 1 \le j \le k-2 \\ \\ L^*(w_{k+1,k+1}) = -\mu^{2k-2} \frac{\Delta_{k-2}(f^{(3-k)})}{\Delta_{k-1}(f^{(1-k)})} \end{cases}$$

Proof. Let $w_{11}, w_{22}, \ldots, w_{kk}, w_{2,2k}, w_{3,2k-1}, \ldots, w_{2k,2}$ be given by (6.8), (6.9) and (6.10). Then for $1 \leq j \leq k-2$ the left hand side of (6.11) is equal to

$$-\frac{1}{\mu^2} \frac{\Delta_{j-1}(f^{(1-j)})\Delta_{j+1}(f^{(-1-j)})A}{\Delta_j(f^{(-j)})^4},$$

where $A = \Delta_{j-1}(f^{(1-j)})\Delta_{j+1}(f^{(-1-j)}) - \Delta_j(f^{(1-j)})\Delta_j(f^{(-1-j)}).$

By applying (6.6) to $g = f^{(-1-j)}$ it follows that $A = -\Delta_j (f^{(-j)})^2$, which proves (6.11) for $1 \le j \le k-2$. The case j = 0 of (6.11) follows immediately from (6.8).

The proofs of (6.12) and 6.13) can be obtained exactly as in the case k = 2 provided the following two identities holds: For j = 0, ..., k - 2:

(6.14)
$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\Delta_{j+1}(f^{(-2-j)})}{\Delta_j(f^{(-j)})} \right) = \frac{\Delta_j(f^{(-1-j)})\Delta_{j+1}(f^{(-1-j)})}{\Delta_j(f^{(-j)})^2}$$

For j = 1, ..., k - 1:

(6.15)
$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\Delta_{j-1}(f^{(2-j)})}{\Delta_j(f^{(-j)})} \right) = \frac{\Delta_{j-1}(f^{(1-j)})\Delta_j(f^{(1-j)})}{\Delta_j(f^{(-j)})^2}$$

However (6.14) follows from (6.7) with $g = f^{(-2-j)}$ after changing j in (6.7) to j + 1. In the same way (6.15) follows from (6.7) with $g = f^{(-j)}$ and j unchanged. It remains to be proved, that $w_{11}, \ldots, w_{kk}, w_{2,2k}, \ldots, w_{2k,2}$ form a solution to (6.4). The proof of the first 2 identities in (6.4) is exactly the same as in the case k = 2. Let us check the next k - 2 identities in (6.4) i.e.

(6.16)
$$\begin{pmatrix} 0 & L(w_{j+1,2k+1-j}) \\ L^*(w_{2k-1-j,j+3}) & 0 \end{pmatrix} + \begin{pmatrix} w_{2+j,2+j} & w_{2+j,2k-j} \\ w_{2k-j,2+j} & w_{2k-j,2k-j} \end{pmatrix}^{-1} = \mu \mathbf{1}_{M_2(\mathcal{D})}$$

for j = 1, ..., k - 2. By (6.11) and the fact that $w_{2+j,2+j} = w_{2k-j,2k-j}$ (cf. (6.8)) we have

$$\begin{pmatrix} w_{2+j,2+j} & w_{2+j,2k-j} \\ w_{2k-j,2+j} & w_{2k-j,2k-j} \end{pmatrix}^{-1} = \begin{pmatrix} \mu \mathbf{1}_{\mathcal{D}} & \beta \\ \gamma & \mu \mathbf{1}_{\mathcal{D}} \end{pmatrix},$$

where

$$\beta = -\mu \frac{w_{2+j,2k-j}}{w_{2+j,2+j}} = \frac{1}{\mu^{2j}} \frac{\Delta_j(f^{(-1-j)})}{\Delta_{j-1}(f^{(1-j)})}$$

and

$$\gamma = -\mu \frac{w_{2k-j,2+j}}{w_{2+j,2+j}} = \mu^{2j+2} \frac{\Delta_j(f^{(1-j)})}{\Delta_{j+1}(f^{(-1-j)})}$$

Hence by (6.12) and (6.13)

$$\beta = -L(w_{j+1,2k-j+1})$$
 and $\gamma = -L^*(w_{2k-1-j,j+3})$

for $j = 1, \ldots, k-2$. This proves (6.16). Observe next that by (6.10) and (6.12)

$$w_{k+1,k+1}(\mu - L(w_{k,k+2})) = \frac{\Delta_{k-1}(f^{(2-k)})\Delta_{k-1}(f^{(-k)})}{\Delta_{k-1}(f^{(1-k)})^2}$$
$$= 1 + \frac{\sigma}{\Delta_{k-1}(f^{(1-k)})^2},$$

where

$$\sigma = \Delta_{k-1}(f^{(2-k)})\Delta_{k-1}(f^{(-k)}) - \Delta_{k-1}(f^{(1-k)})^2$$

By (6.6) and the assumptions (iii) in lemma 6.3

$$\sigma = \Delta_{k-2}(f^{(2-k)})\Delta_k(f^{(-k)}) = 0.$$

Hence $w_{k+1,k+1}(\mu - L(w_{k,k+2})) = 1$, which proves the last equality in (6.4). This completes the proof of lemma 6.3.

Lemma 6.4. Let $k \in \mathbb{N}, k \geq 2$ and let $\alpha_j(s), \gamma_j(s)$ for $j = 1, \ldots, k$ and $0 < |s| < \frac{1}{e}$ be as in lemma 5.2. Let $\mu \in \mathbb{C}, |\mu| > \sqrt{e}$, put $s = \frac{1}{k}\mu^{-2}$ and

(6.17)
$$\begin{cases} f(x) = \frac{1}{\mu} \left(\sum_{\nu=1}^{k} \gamma_{\nu}(s) \mathrm{e}^{k\alpha_{\nu}(s)x} \right), & x \in \mathbb{R} \\ f^{(-j)}(x) = \frac{1}{\mu k^{j}} \left(\sum_{\nu=1}^{k} \frac{\gamma_{\nu}(s)}{\alpha_{\nu}(s)^{j}} \mathrm{e}^{k\alpha_{\nu}(s)x} \right), & x \in \mathbb{R}, j = 1, \dots, k \end{cases}$$

Then

(i) $(f^{(-j)})_{j=1}^k$ are successive antiderivatives of f. Moreover

(6.18)
$$\begin{cases} f^{(-j)}(1) = 0, & 1 \le j \le k-1 \\ f^{(-k)}(1) = \mu^{2k-1} \end{cases}$$

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and

(6.19)
$$\begin{cases} f(0) = \mu^{-1} \\ f^{(j)}(0) = 0, \quad 1 \le j \le k - 1 \end{cases}.$$

(ii) The following asymptotic formulas holds for $|\mu| \to \infty$

(6.20)
$$\begin{cases} f^{(-k)}(x) = \mu^{2k-1} + \mathcal{O}(\mu^{-1}) \\ f^{(-j)}(x) = \frac{1}{j!}(x-1)^{j}\mu^{-1} + \mathcal{O}(\mu^{-2k-1}), & 1 \le j \le k-1 \\ f(x) = \mu^{-1} + \mathcal{O}(\mu^{-2k-1}) & , \\ f^{(j)}(x) = \frac{1}{j!}x^{j}\mu^{-2k-1} + \mathcal{O}(\mu^{-4k-1}), & 1 \le j \le k-1 \\ f^{(k)}(x) = \mu^{-2k-1} + \mathcal{O}(\mu^{-4k-1}) & \end{cases}$$

where the error estimates holds uniformly in x on compact subsets of \mathbb{R} . (iii) There exists a $\mu_0 \geq \sqrt{e}$, such that the restriction of f to [0,1] satisfies all the conditions in lemma 6.3, when $|\mu| > \mu_0$.

Proof. From the proof of [5, Prop. 4.2], we know that $\alpha_j(s)$ and $\gamma_j(s)$ are analytic functions of $s \in B(0, \frac{1}{e})$. Moreover by [4, Prop. 4.1]

(6.21)
$$\begin{cases} \sum_{\nu=1}^{k} \gamma_{\nu}(s) = 1\\ \sum_{\nu=1}^{k} \gamma_{\nu}(s) \alpha_{\mu}(s)^{j} = 1, \quad j = 1, \dots, k-1 \end{cases}$$

Moreover, since $\alpha_j(s) = \rho(e^{i\frac{2\pi j}{k}s})$, where ρ satisfies

$$\rho(w) \mathrm{e}^{-\rho(w)} = w \text{ for } |w| < \frac{1}{\mathrm{e}}$$

we have $(\alpha_{\nu}(s)e^{-\alpha_{\nu}(s)})^k = s^k$ and therefore

(6.22)
$$e^{k\alpha_{\nu}(s)} = \frac{s^k}{(\alpha_{\nu}(s))^k}$$

for $\nu = 1, \ldots, k$. Having (6.21) and (6.22) in mind, the proof of (i) and (ii) in lemma 6.4 is now a routine generalization of the proof of lemma 5.5. Concerning (iii) in lemma 6.4, we have

(6.23)
$$\begin{cases} \Delta_j(f^{(-j)}) = \sigma(j)\mu^{-j-1} + \mathcal{O}(\mu^{-2k-j-1}), & 0, \dots, k-1, \\ \text{where } \sigma(j) = 1 \text{ for } j = 0, 3 \pmod{4} \\ \text{and } \sigma(j) = -1 \text{ for } j = 1, 2 \pmod{4} \end{cases}$$

because the leading term in the determinant $\Delta_j(f^{(-j)})$ comes from the antidiagonal, i.e.

$$\Delta_{j}(f^{(-j)}) = \begin{vmatrix} 0 & \dots & 0 & f \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \vdots \\ f & 0 & \dots & 0 \end{vmatrix} + \mathcal{O}(\mu^{-2k-j-1}) = \sigma(j)f^{j+1} + \mathcal{O}(\mu^{-2k-j-1})$$

since the matrix in question has size j+1. Hence $\Delta_j(f^{(-j)})(x) \neq 0$ for $x \in [0, 1]$ and $0 \leq j \leq k-1$, when $|\mu|$ is sufficiently large. Moreover $\Delta_k(f^{(-k)}) = 0$ for $x \in [0, 1]$, because in analogy with (5.29), $\Delta_k(f^{(-k)}(x))$ is the determinant of the $(k+1) \times (k+1)$ matrix

$$F = (f^{(i+j-k)})_{i,j=0,...,k}$$

which has the factorization $F = ADA^t$, where A is the $(k+1) \times k$ matrix with entries

$$a_{il} = (k\alpha_l(s))^i, \quad i = 0, \dots, k, \quad l = 1, \dots, k$$

and D is the $k \times k$ diagonal matrix, with diagonal entries

$$d_{ll} = \frac{\gamma_l(s)}{(k\alpha_l(s))^k} e^{k\alpha_l(s)}, \quad l = 1, \dots, k.$$

Proof of Theorem 5.1 in the general case. Let μ_0 be as in lemma 6.4, let $\mu \in \mathbb{C}, |\mu| > \mu_0$ and put $s = \frac{1}{k}\mu^{-2}$. Put as before

$$f(x) = \frac{1}{\mu} \left(\sum_{\nu=1}^{k} \gamma_j(s) \mathrm{e}^{k\alpha_j(s)x} \right)$$

for $x \in [0, 1]$, and define $w_{11}, w_{22}, \ldots, w_{k,k}, w_{2,2k}, w_{3,2k-1}, \ldots, w_{2k,2}$ by (6.8), (6.9) and (6.10), and put all other entries of $w \in M_{2k}(\mathcal{D})$ equal to 0. Then by lemma 6.4, (6.4) holds, and therefore

$$\mathcal{R}^{M_{2k}(\mathcal{D})}_{\tilde{T}}(w) + w^{-1} = \mu \mathbf{1}_{M_{2k}(\mathcal{D})}.$$

Let $\delta > 0$ be chosen according to lemma 2.1. If we can find a $\mu_1 \ge \max\{\mu_0, \frac{1}{\delta}\}$, such that

$$(6.24) |\mu| \ge \mu_1 \Rightarrow ||w|| < \delta$$

then $w = E_{M_{2k}(\mathcal{D})}((\tilde{\mu} - \tilde{T})^{-1})$. In particular

(6.25)
$$f = w_{11} = \mu^{-1} E_{\mathcal{D}}((1 - \mu^{-2k} (T^*)^k T^k)^{-1}),$$

and the proof of theorem 5.1 for $k \ge 2$ can be completed exactly as in the case k = 2. By (6.23)

(6.26)
$$\begin{cases} \Delta_j(f^{(-j)}) = \mathcal{O}(\mu^{-j-1}), & 0 \le j \le k-1 \\ \frac{1}{\Delta_j(f^{(-j)})} = \mathcal{O}(\mu^{j+1}), & 0 \le j \le k-1 \end{cases}$$

uniformly in $x \in [0, 1]$ for $|\mu| \to \infty$. We claim that

(6.27)
$$\begin{cases} \Delta_j(f^{(-j-1)}) = \mathcal{O}(\mu^{-j-1}), & 0 \le j \le k-2 \\ \Delta_{k-1}(f^{(-k)}) = \mathcal{O}(\mu^k) \\ \Delta_j(f^{(1-j)}) = \mathcal{O}(\mu^{-j-2k-1}), & 0 \le j \le k-2 \\ \Delta_{k-1}(f^{(2-k)}) = \mathcal{O}(\mu^{-3k}) \end{cases}$$

Recall by definition 6.1 that

$$\Delta_j(g) = \det\left((g^{(k+l)})_{k,l=0,\ldots,j}\right).$$

Hence for $0 \leq j \leq k-2$, $\Delta_j(f^{(-j-1)})$ is the determinant of a $(j+1) \times (j+1)$ matrix, where each entry is equal to one of the functions $f^{(-j-1)}, f^{(-j)}, \ldots, f^{(j-1)}$. By (6.20) all these functions are of order $\mathcal{O}(\mu^{-1})$ as $|\mu| \to \infty$. Hence

$$\Delta_j(f^{(-j-1)}) = \mathcal{O}(\mu^{-j-1})$$

proving the first estimate in (6.27). By the same argument, $\Delta_{k-1}(f^{(-k)})$ is the determinant of a $k \times k$ matrix for which the upper left entry is of the order $\mathcal{O}(\mu^{2k-1})$ and all the other entries are of order $\mathcal{O}(\mu^{-1})$. Hence $\Delta_{k-1}(f^{(-k)}) = \mathcal{O}(\mu^{2k-1}(\mu^{-1})^{k-1}) = \mathcal{O}(\mu^k)$. Let $0 \leq j \leq k-1$. Then $\Delta_j(f^{(1-j)})$ is by (6.20) a determinant of a $(j+1) \times (j+1)$ matrix $M = (m_{k,l})_{k,l=0,\dots,j}$ for which

$$\begin{cases} m_{k,l} = \mathcal{O}(\mu^{-1}) & \text{when } k+l < 0\\ m_{k,l} = \mathcal{O}(\mu^{-2k-1}) & \text{when } k+l \ge 0 \end{cases}$$

Hence for any permutation π of $\{0, 1, \ldots, k\}$ the product

$$m_{0\pi(0)}m_{1\pi(1)}\cdots m_{j\pi(j)}$$

contains at least one factor of order $\mathcal{O}(\mu^{-2k-1})$. Therefore

$$\Delta_j(f^{(1-j)}) = \det(M) = \sum_{\pi \in S_{j+1}} (-1)^{\operatorname{sign}(\pi)} m_{0\pi(0)} m_{1\pi(1)} \cdots m_{k\pi(k)}$$

is of order $\mathcal{O}(\mu^{-2k-1}(\mu^{-1})^j) = \mathcal{O}(\mu^{-2k-j-1})$. This proves the last two estimates in (6.27). Clearly all estimates holds uniformly in $x \in [0, 1]$. Combining (6.8), (6.9), (6.10) and (6.27), we get

$$\begin{cases} w_{l,l} = \mathcal{O}(\mu^{-1}), & 1 \le l \le 2k \\ w_{j+2,2k-j} = \mathcal{O}(\mu^{-2j-3}), & 0 \le j \le k-2 \\ w_{2k-j,j+2} = \mathcal{O}(\mu^{2j+1-2k}), & 0 \le j \le k-2 \end{cases}$$

In particular all the entries of w are of size $\mathcal{O}(\mu^{-1})$ as $|\mu| \to \infty$ uniformly in $x \in [0, 1]$. Hence there exists $\mu_1 \ge \max\{\mu_0, \frac{1}{\delta}\}$ such that (6.24) holds. Hence by (6.25) we have for $|s| < \frac{1}{k}\mu_1^{-2}$,

$$\sum_{k=0}^{\infty} (ks)^{nk} E_{\mathcal{D}}(((T^*)^k T^k)^n)(x) = \sum_{\nu=1}^{\infty} \gamma_j(s) e^{k\alpha_j(s)x}, \quad x \in [0,1].$$

Now Theorem 5.1 follows from lemma 5.2 and [5, remark 4.3] as in the case k = 2.

APPENDIX A. DETERMINANT-IDENTITIES ON HANKEL-MATRICES

We need the following lemma on Hankel-determinants.

Lemma A.1. Let $a_{-(n-1)}, a_{-(n-2)}, \ldots, a_{n-1}, a_n \in \mathbb{C}$ for some $n \in \mathbb{N}$. Then (a)

$$\begin{vmatrix} a_{-(n-3)} & a_{-(n-4)} & \ddots & a_{0} \\ a_{-(n-4)} & \ddots & \ddots & \ddots & a_{n-4} \\ a_{-(n-4)} & \ddots & \ddots & \ddots & a_{n-4} \\ a_{0} & \ddots & a_{n-4} & a_{n-3} \end{vmatrix} \begin{vmatrix} a_{-(n-2)} & a_{-(n-3)} & a_{-(n-4)} & \ddots & \ddots & \ddots & a_{n-4} \\ a_{-(n-3)} & a_{-(n-4)} & \ddots & \ddots & \ddots & a_{n-4} & a_{n-3} \\ a_{-(n-4)} & \ddots & \ddots & \ddots & a_{n-4} & a_{n-3} \\ a_{0} & \ddots & a_{n-4} & a_{n-3} & a_{n-2} \\ a_{0} & \ddots & a_{n-4} & a_{n-3} & a_{n-2} \\ a_{0} & \ddots & a_{n-4} & a_{n-3} & a_{n-2} \\ a_{0} & \ddots & a_{n-4} & a_{n-3} & a_{n-2} \\ a_{0} & \ddots & a_{n-4} & a_{n-3} \end{vmatrix} \begin{vmatrix} a_{-(n-4)} & \ddots & a_{1} \\ a_{-(n-4)} & \ddots & \ddots \\ \vdots & \ddots & \vdots & a_{n-4} \\ a_{-1} & \ddots & a_{n-4} \\ a_{-1} & \ddots & a_{n-4} \\ a_{-1} & a_{n-4} & a_{n-3} \end{vmatrix} \begin{vmatrix} a_{-(n-4)} & \ddots & a_{1} \\ a_{-(n-4)} & \ddots & \ddots \\ \vdots & \ddots & \vdots & a_{n-2} \\ a_{1} & \ddots & a_{n-2} \\ a_{0} & \vdots & a_{n-3} \\ a$$

(b)

$$\begin{vmatrix} a_{-(n-2)} & a_{-(n-3)} & \ddots & a_{1} \\ a_{-(n-3)} & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & a_{n-1} \\ a_{1} & \ddots & a_{n-1} & a_{n} \end{vmatrix} \begin{vmatrix} a_{-(n-2)} & a_{-(n-3)} & \ddots & a_{0} \\ a_{-(n-3)} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-3} \\ a_{0} & \ddots & a_{n-3} & a_{n-2} \end{vmatrix}$$
$$= \begin{vmatrix} a_{-(n-1)} & a_{-(n-2)} & \ddots & a_{0} \\ a_{-(n-2)} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-3} \\ a_{-1} & a_{2} & \cdots & a_{n} \end{vmatrix} \begin{vmatrix} a_{-(n-4)} & \ddots & a_{1} \\ a_{-(n-4)} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-2} \\ a_{1} & a_{2} & \cdots & a_{n} \end{vmatrix} \begin{vmatrix} a_{-(n-3)} & a_{-(n-4)} & \ddots & a_{1} \\ a_{-(n-4)} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-2} \\ a_{1} & a_{2} & \cdots & a_{n} \end{vmatrix} \begin{vmatrix} a_{-(n-3)} & a_{-(n-4)} & \ddots & a_{1} \\ a_{-(n-4)} & \ddots & \ddots & \vdots \\ a_{n-2} & a_{n-1} & a_{n-2} & a_{n-1} \end{vmatrix}$$

Proof. To prove (a) we actually prove the more general equation

(A.1)
$$\begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2,n-1} \\ a_{32} & a_{33} & \cdots & a_{3,n-1} \\ \vdots & \vdots & & \vdots \\ a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} \\ a_{21} & a_{22} & \cdots & a_{2,n-1} \\ \vdots & \vdots & & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{vmatrix} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2,n} \\ a_{32} & a_{33} & \cdots & a_{3,n} \\ \vdots & & \vdots & & \vdots \\ a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{vmatrix} \\ - \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1,n} \\ a_{22} & a_{23} & \cdots & a_{2,n} \\ \vdots & & & \vdots \\ a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n} \end{vmatrix} \begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2,n-1} \\ a_{31} & a_{32} & \cdots & a_{3,n-1} \\ \vdots & & & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} \end{vmatrix}$$

for $a_{ij} \in \mathbb{C}$ and $i, j \in \{1, \ldots, n\}$.

We first add some zero terms to the left-hand side (LHS) of (A.1).

$$LHS = \begin{vmatrix} a_{22} & \cdots & a_{2,n-1} \\ \vdots & \vdots \\ a_{n-1,2} & \cdots & a_{n-1,n-1} \end{vmatrix} \begin{vmatrix} a_{11} & \cdots & a_{1,n} \\ \vdots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{vmatrix}$$
$$+ \sum_{k=2}^{n-1} \begin{vmatrix} a_{21} & \cdots & a_{2,k-1} & a_{2,k+1} & \cdots & a_{2,n-1} \\ a_{31} & \cdots & a_{3,k-1} & a_{3,k+1} & \cdots & a_{3,n-1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & \cdots & a_{n-1,n-1} \end{vmatrix} \begin{vmatrix} a_{12} & \cdots & a_{1,k} & a_{1,k} & a_{1,k+1} & \cdots & a_{2,n-1} \\ a_{22} & \cdots & a_{2,k} & a_{2,k} & a_{2,k+1} & \cdots & a_{3,n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n,2} & \cdots & a_{n,k} & a_{n,k} & a_{n,k+1} & \cdots & a_{n-1,n-1} \end{vmatrix}$$

We note that the last matrix in the sum is zero because coloumn k-1 and k are equal. Now we expand LHS after the k'th coloumn of the second matrix in the k'th addent. We get

$$LHS = \sum_{j=1}^{n} (-1)^{1+j} a_{j,1} \begin{vmatrix} a_{22} & \cdots & a_{2,n-1} \\ \vdots & \vdots \\ a_{n-1,2} & \cdots & a_{n-1,n-1} \end{vmatrix} \begin{vmatrix} a_{12} & \cdots & a_{1,n} \\ \vdots & \vdots \\ a_{j-1,2} & \cdots & a_{j-1,n} \\ a_{j+1,2} & \cdots & a_{j+1,n} \\ \vdots & \vdots \\ a_{n,2} & \cdots & a_{n,n} \end{vmatrix} + \sum_{k=2}^{n-1} \sum_{j=1}^{n} (-1)^{k+j} a_{j,k} \begin{vmatrix} a_{21} & \cdots & a_{2,k-1} & a_{2,k+1} & \cdots & a_{2,n-1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & \cdots & a_{n-1,n-1} \end{vmatrix} \begin{vmatrix} a_{12} & \cdots & a_{1,n} \\ \vdots & \vdots \\ a_{j-1,2} & \cdots & a_{j-1,n} \\ a_{j+1,2} & \cdots & a_{j+1,n} \\ \vdots & \vdots \\ a_{n,2} & \cdots & a_{n,n} \end{vmatrix}$$

where j = 1 and j = n means leave out row 1 and n respectively. Switching the indices we have

(A.2) LHS =
$$\sum_{j=1}^{n} \begin{vmatrix} a_{12} & \cdots & a_{1,n} \\ \vdots & \vdots \\ a_{j-1,2} & \cdots & a_{j-1,n} \\ a_{j+1,2} & \cdots & a_{j+1,n} \\ \vdots & \vdots \\ a_{n,2} & \cdots & a_{n,n} \end{vmatrix} \begin{pmatrix} (-1)^{1+j}a_{j,1} \begin{vmatrix} a_{22} & \cdots & a_{2,n-1} \\ \vdots & \vdots \\ a_{n-1,2} & \cdots & a_{n-1,n-1} \end{vmatrix}$$

 $\sum_{k=2}^{n-1} (-1)^{k+j}a_{j,k} \begin{vmatrix} a_{21} & \cdots & a_{2,k-1} & a_{2,k+1} & \cdots & a_{2,n-1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & \cdots & a_{n-1,n-1} \end{vmatrix}$

But the parenthesis on the right-hand side is exactly expansion along the j'th row of the following determinants

(A.3)
$$\begin{cases} \begin{vmatrix} a_{11} & \cdots & a_{1,n-1} \\ \vdots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n-1} \end{vmatrix}, \quad j = 1 \\ \begin{vmatrix} a_{21} & \cdots & a_{2,n-1} \\ \vdots & \vdots \\ a_{j,1} & \cdots & a_{j,n-1} \\ \vdots & \vdots \\ a_{n,1} & \cdots & a_{n,n-1} \end{vmatrix} = 0, \quad 2 \le j \le n-1 \\ \begin{vmatrix} a_{21} & \cdots & a_{2,n-1} \\ \vdots & \vdots \\ a_{n,1} & \cdots & a_{n,n-1} \end{vmatrix}, \quad j = n.$$

Combining (A.2) and (A.3) we obtain the right-hand side of (A.1) and thus also the proof of (a).

To prove (b) we prove the more general equation

$$(A.4) \qquad \begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2,n} \\ a_{31} & a_{32} & \cdots & a_{3,n} \\ a_{41} & a_{42} & \cdots & a_{4,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} \end{vmatrix} \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1,n} \\ a_{22} & a_{23} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n} \end{vmatrix} \\ = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n} \\ a_{21} & a_{22} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots \\ a_{n-1,1} & a_{n+1,2} & \cdots & a_{n+1,n} \\ a_{21} & a_{22} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots \\ a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{vmatrix} \\ = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n} \\ a_{21} & a_{22} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n} \\ a_{21} & a_{22} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n,n} \end{vmatrix} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2,n} \\ a_{32} & a_{33} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots \\ a_{n-2} & a_{n,3} & \cdots & a_{n,n} \end{vmatrix}$$

for $a_{ij} \in \mathbb{C}$, $i \in \{1, \ldots, n+1\}$ and $j \in \{1, \ldots, n\}$. We remark that Hankelmatrices are symmetric and for these (A.4) reduces to (b). Observe that for $k \in \{2, \ldots, n\}$ we have

$$0 = (-1)^{k} \begin{vmatrix} a_{1,k} & a_{11} & a_{12} & \cdots & a_{1,n} \\ a_{2,k} & a_{12} & a_{22} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,k} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \\ a_{n+1,k} & a_{n+1,2} & a_{n+1,3} & \cdots & a_{n+1,n} \end{vmatrix}$$
$$= (-1)^{k} \sum_{j=1}^{n+1} a_{j,k} (-1)^{j+1} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n} \\ \vdots & \vdots & \vdots \\ a_{j-1,1} & a_{j-1,2} & \cdots & a_{j-1,n} \\ a_{j+1,1} & a_{j+1,2} & \cdots & a_{j+1,n} \\ \vdots & \vdots & \vdots \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} \end{vmatrix},$$

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where the j = 1 and j = n + 1 are interpreted as remove the 1st and (n + 1)th coloumn respectively. Thus also

$$0 = \sum_{k=2}^{n} \begin{vmatrix} a_{22} & \cdots & a_{2,k-1} & a_{2,k+1} & \cdots & a_{2,n} \\ a_{32} & \cdots & a_{3,k-1} & a_{3,k+1} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n-1,2} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & \cdots & a_{n-1,n} \end{vmatrix}$$
$$\cdot \left((-1)^{k} \sum_{j=1}^{n+1} a_{j,k} (-1)^{j+1} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n} \\ \vdots & \vdots & \vdots \\ a_{j-1,1} & a_{j-1,2} & \cdots & a_{j-1,n} \\ a_{j+1,1} & a_{j+1,2} & \cdots & a_{j+1,n} \\ \vdots & \vdots & \vdots \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} \end{vmatrix} \right)$$

Switching the indices we have

$$(A.5) \quad 0 = \sum_{j=1}^{n+1} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n} \\ \vdots & \vdots & \vdots \\ a_{j-1,1} & a_{j-1,2} & \cdots & a_{j-1,n} \\ a_{j+1,1} & a_{j+1,2} & \cdots & a_{j+1,n} \\ \vdots & \vdots & \vdots \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} \end{vmatrix} \\ \cdot \left(\sum_{k=2}^{n} (-1)^{k+j-1} a_{j,k} \begin{vmatrix} a_{22} & \cdots & a_{2,k-1} & a_{2,k+1} & \cdots & a_{2,n} \\ a_{32} & \cdots & a_{3,k-1} & a_{3,k+1} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n-1,2} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & \cdots & a_{n-1,n} \end{vmatrix} \right)$$

The parenthesis of (A.5) is expansion along the j^{th} row of the following expression except for j = n + 1 where we expand along the n^{th} row.

(A.6)
$$\begin{cases} \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1,n} \\ a_{22} & a_{23} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n} \end{vmatrix}, \qquad j = 1 \\ 0, \qquad \qquad j \in \{2, \dots, n-1\} \\ \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2,n} \\ a_{32} & a_{33} & \cdots & a_{3,n} \end{vmatrix}, \qquad \qquad j = n \\ \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots \\ a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{vmatrix}, \qquad \qquad j = n \\ - \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots \\ a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n} \\ a_{n+1,2} & a_{n+1,3} & \cdots & a_{n+1,n} \end{vmatrix} \qquad j = n + 1.$$

Combining (A.5) and (A.6) we obtain (A.4) and this finishes the proof of (b). \Box

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