Separability of quantum states and the violation of Bell-type inequalities

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Abstract

In contrast to the wide-spread opinion that any separable quantum state satisfies every classical probabilistic constraint, we present a simple example where a separable quantum state does not satisfy the original Bell inequality although the latter inequality, in its perfect correlation form, is valid for all joint classical measurements.

In a very general setting, we discuss inequalities for joint experiments upon a bipartite quantum system. For any separable quantum state, we derive quantum analogues of the original Bell inequality and specify the conditions sufficient for a separable state to satisfy the original Bell inequality. We introduce the extended CHSH inequality and prove that, for any separable quantum state, this inequality holds for a variety of linear combinations.

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1 Introduction

The relation between non-separability and the violation of Bell-type inequalities is discussed in many papers (see, for example, the review [1] and references therein). It has been argued that any separable quantum state satisfies every Bell-type inequality.

A Bell-type inequality is usually viewed as any constraint on averages or probabilities arising under the description of joint experiments in the classical probabilistic frame. In this paper, we consider a simple example where, for a joint experiment upon the two-qubit system in a separable

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state, the original Bell inequality [2-4], in its perfect correlation form, is violated. The latter inequality is, however, valid for all joint classical measurements[1].

In a very general setting, we analyze inequalities arising for quantum "locally realistic" [5] joint experiments on a bipartite quantum system. For any separable state, we derive quantum analogues of the original Bell inequality and specify the conditions where a separable state satisfies the original Bell inequality, in its perfect correlation or anti-correlation forms. These sufficient conditions include Bell's correlation restrictions as particular cases.

We introduce the extended CHSH inequality for a linear combination of mean values and prove that, for any separable quantum state, this inequality is valid for a variety of linear combinations. The latter fact may be used for distinguishing between separable and non-separable quantum states via computer processing of linear combinations of statistical averages.

2 Violation of Bell's inequality for a separable state

Consider a bipartite quantum system $\mathcal{S}_q + \mathcal{S}_q$, described in terms of the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2$.

Following the presentation in [6], page 156, denote by

$$J^{(\theta)} = \{|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\}\cos 2\theta + \{|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\}\sin 2\theta \tag{1}$$

a self-adjoint operator on $\mathbb{C}^2$ with eigenvalues $\lambda^{(\theta)}_{1,2} = \pm 1$. Here, θ is a real valued parameter representing from the physical point of view an angle from some axis (say, z-axis) and by the symbols $|\uparrow\rangle$ and $|\downarrow\rangle$ we denote eigenvectors of the operator $J^{(0)}$, corresponding to the eigenvalues $(+1)$ and (-1), respectively. Formally, the operator $J^{(\theta)}$ represents the spin operator $S_z \cos 2\theta + S_x \sin 2\theta$.

Under a joint experiment, constituting a measurement of a quantum observable $J^{(\theta_1)} \otimes J^{(\theta_2)}$, $\forall \theta_1, \theta_2$, and performed upon the two-qubit system in a state ρ, the quantum average

$$\langle J^{(\theta_1)} \otimes J^{(\theta_2)} \rangle_\rho := \mathrm{tr}[\rho\{J^{(\theta_1)} \otimes J^{(\theta_2)}\}] \tag{2}$$

represents the expectation value

$$\langle \lambda^{(\theta_1)} \lambda^{(\theta_2)} \rangle_\rho = \langle J^{(\theta_1)} \otimes J^{(\theta_2)} \rangle_\rho \tag{3}$$

of the product $\lambda^{(\theta_1)}\lambda^{(\theta_2)}$ of the observed outcomes.

For three different joint experiments upon $\mathcal{S}_q + \mathcal{S}_q$, constituting measurements of quantum observables

$$J^{(\theta_a)} \otimes J^{(\theta_b)}, \qquad J^{(\theta_a)} \otimes J^{(\theta_c)}, \qquad J^{(\theta_b)} \otimes J^{(\theta_c)}, \tag{4}$$

respectively, consider the corresponding expectation values

$$\langle \lambda^{(\theta_a)} \lambda^{(\theta_b)} \rangle_\rho, \qquad \langle \lambda^{(\theta_a)} \lambda^{(\theta_c)} \rangle_\rho, \qquad \langle \lambda^{(\theta_b)} \lambda^{(\theta_c)} \rangle_\rho, \tag{5}$$

for the case where the two-qubit is initially in the same state ρ. (Here, a, b, c are indices, specifying different θ.)

Take a separable initial quantum state[2]

$$\rho_0 = \frac{1}{2}\{|\uparrow><\uparrow| \otimes |\downarrow><\downarrow| + |\downarrow><\downarrow| \otimes |\uparrow><\uparrow|\}. \tag{6}$$

Then the corresponding expectation values (5) are given by:

$$\begin{aligned} \langle \lambda^{(\theta_a)} \lambda^{(\theta_b)} \rangle_{\rho_0} &= \mathrm{tr}[\rho_0\{J^{(\theta_a)} \otimes J^{(\theta_b)}\}] = -\cos 2\theta_a \ \cos 2\theta_b, \\ \langle \lambda^{(\theta_a)} \lambda^{(\theta_c)} \rangle_{\rho_0} &= \mathrm{tr}[\rho_0\{J^{(\theta_a)} \otimes J^{(\theta_c)}\}] = -\cos 2\theta_a \ \cos 2\theta_c, \\ \langle \lambda^{(\theta_b)} \lambda^{(\theta_c)} \rangle_{\rho_0} &= \mathrm{tr}[\rho_0\{J^{(\theta_b)} \otimes J^{(\theta_c)}\}] = -\cos 2\theta_b \ \cos 2\theta_c. \end{aligned} \tag{7}$$

[1] See Appendix.

[2] Since quantum sub-systems are identical, we take a symmetrized initial density operator.

For the three joint experiments considered, the original Bell inequality, in its perfect correlation form[3],

$$\left|\langle\lambda^{(\theta_a)}\lambda^{(\theta_b)}\rangle_{\rho_0} - \langle\lambda^{(\theta_a)}\lambda^{(\theta_c)}\rangle_{\rho_0}\right| \leq 1 - \langle\lambda^{(\theta_b)}\lambda^{(\theta_c)}\rangle_{\rho_0} \tag{8}$$

reads

$$|\cos 2\theta_a \cos 2\theta_b - \cos 2\theta_a \cos 2\theta_c)| \leq 1 + \cos 2\theta_b \cos 2\theta_c. \tag{9}$$

For example, let $\theta_a = 0$, $\theta_b = \pi/6$, $\theta_c = \pi/3$, then

$$\cos 2\theta_a = 1, \qquad \cos 2\theta_b = 1/2, \qquad \cos 2\theta_c = -1/2. \tag{10}$$

Substituting (10) into (9), we derive the obvious violation.

It is also easy to verify the violation of (8) if the expectation values, standing in this inequality, are expressed via the symmetrized tensor products (for details, see section 3.2):

$$\begin{aligned}
\langle\lambda^{(\theta_a)}\lambda^{(\theta_b)}\rangle_{\rho_0} &= \mathrm{tr}[\rho_0\{J^{(\theta_a)} \otimes J^{(\theta_b)}\}_{sym}], \\
\langle\lambda^{(\theta_a)}\lambda^{(\theta_c)}\rangle_{\rho_0} &= \mathrm{tr}[\rho_0\{J^{(\theta_a)} \otimes J^{(\theta_c)}\}_{sym}], \\
\langle\lambda^{(\theta_b)}\lambda^{(\theta_c)}\rangle_{\rho_0} &= \mathrm{tr}[\rho_0\{J^{(\theta_b)} \otimes J^{(\theta_c)}\}_{sym}],
\end{aligned} \tag{11}$$

where we introduce the notation

$$\{V_1 \otimes V_2\}_{sym} := \frac{1}{2}\{V_1 \otimes V_2 + V_2 \otimes V_1\}, \tag{12}$$

for any operators V_1, V_2.

The above simple example shows that the statement that any separable state satisfies every classical constraint is false.

For the example considered, let us also check the validity of the original CHSH inequality [7].

Suppose that we have four different joint experiments, constituting measurements of quantum observables

$$J^{(\theta_a)} \otimes J^{(\theta_b)}, \quad J^{(\theta_c)} \otimes J^{(\theta_b)}, \quad J^{(\theta_c)} \otimes J^{(\theta_d)} \quad J^{(\theta_a)} \otimes J^{(\theta_d)}, \tag{13}$$

and performed on the two-qubit in a state ρ_0. Here the indices a, b, c, d specify different θ.

For these four joint experiments, consider the left-hand side of the original CHSH inequality:

$$\left|\langle\lambda^{(\theta_a)}\lambda^{(\theta_b)}\rangle_{\rho_0} + \langle\lambda^{(\theta_c)}\lambda^{(\theta_b)}\rangle_{\rho_0} + \langle\lambda^{(\theta_c)}\lambda^{(\theta_d)}\rangle_{\rho_0} - \langle\lambda^{(\theta_a)}\lambda^{(\theta_d)}\rangle_{\rho_0}\right| \leq 2. \tag{14}$$

From (7) it follows,

$$\begin{aligned}
&\left|\langle\lambda^{(\theta_a)}\lambda^{(\theta_b)}\rangle_{\rho_0} + \langle\lambda^{(\theta_c)}\lambda^{(\theta_b)}\rangle_{\rho_0} + \langle\lambda^{(\theta_c)}\lambda^{(\theta_d)}\rangle_{\rho_0} - \langle\lambda^{(\theta_a)}\lambda^{(\theta_d)}\rangle_{\rho_0}\right| \\
&= |\cos 2\theta_a \cos 2\theta_b + \cos 2\theta_c \cos 2\theta_b + \cos 2\theta_c \cos 2\theta_d - \cos 2\theta_a \cos 2\theta_d|.
\end{aligned} \tag{15}$$

Due to the inequality

$$|x - y| \leq 1 - xy, \tag{16}$$

which is valid, for any $|x| \leq 1$, $|y| \leq 1$, we further derive for the right-hand side of (15):

$$\begin{aligned}
&|\cos 2\theta_a \cos 2\theta_b + \cos 2\theta_c \cos 2\theta_b + \cos 2\theta_c cos2\theta_d - \cos 2\theta_a \cos 2\theta_d| \\
&\leq |\cos 2\theta_a \cos 2\theta_b + \cos 2\theta_c \cos 2\theta_b| + |\cos 2\theta_c \cos 2\theta_d - \cos 2\theta_a \cos 2\theta_d| \\
&\leq 1 + \cos 2\theta_a \cos 2\theta_c + 1 - \cos 2\theta_a \cos 2\theta_c \\
&\leq 2.
\end{aligned} \tag{17}$$

Hence, *in the considered example, the original CHSH inequality holds although the original Bell inequality, in its perfect correlation form, is violated.*

Both, the original CHSH inequality and the original Bell inequality (in its perfect correlation form), represent the probabilistic constraints valid under all joint classical measurements.

[3] See [2-4] and also [6], page 163, and the formula (A6) in Appendix.

3 Quantum inequalities for a separable state

In a very general setting, let us analyze inequalities arising under quantum "locally realistic" (see [5]) joint experiments of the Alice/Bob type, performed on a bipartite quantum system in a separable state.

3.1 General case

Let $\mathcal{S}_q^{(1)}+\mathcal{S}_q^{(2)}$ be a bipartite quantum system, described in terms of a separable complex Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Consider two joint experiments on $\mathcal{S}_q^{(1)}+\mathcal{S}_q^{(2)}$, each with real valued outcomes in a set Λ_1 and in a set Λ_2, and described by the POV measures:

$$\begin{aligned} M^{(\alpha,\beta_1)}(B_1 \times B_2) &= M_1^{(\alpha)}(B_1) \otimes M_2^{(\beta_1)}(B_2), \\ M^{(\alpha,\beta_2)}(B_1 \times B_2) &= M_1^{(\alpha)}(B_1) \otimes M_2^{(\beta_2)}(B_2), \end{aligned} \tag{18}$$

for any subset B_1 of Λ_1 and any subset B_2 of Λ_2.

In (18), the parameters $\alpha, \beta \in \Gamma$ are of any nature and characterize set-ups of the corresponding (marginal) experiments with outcomes in Λ_1 and in Λ_2, respectively. (Λ_1 may be thought as a set of outcomes on the "side" of Alice and Λ_2 as a set of outcomes on the "side" of Bob.)

For simplicity, we suppose that the absolute value of each observed outcome is bounded, that is:

$$\Lambda_1 = \{\lambda_1 \in \mathbb{R} : |\lambda_1| \leq \mathrm{C}_1\}, \qquad \Lambda_2 = \{\lambda_2 \in \mathbb{R} : |\lambda_2| \leq \mathrm{C}_2\}, \tag{19}$$

with some $C_1 > 0$, $C_2 > 0$.

For each of the joint experiments (18), performed on a bipartite quantum system $\mathcal{S}_q^{(1)}+\mathcal{S}_q^{(2)}$ in an initial state ρ :

- the formula
$$\mathrm{tr}[\rho\{M_1^{(\alpha)}(B_1) \otimes M_2^{(\beta_n)}(B_2)\}], \qquad n = 1, 2, \tag{20}$$
represents the joint probability of an outcome λ_1 belonging to a subset $B_1 \subseteq \Lambda_1$ and an outcome λ_2 belonging to a subset $B_2 \subseteq \Lambda_2$, that is, of a compound outcome $(\lambda_1, \lambda_2) \in B_1 \times B_2$;

- the formula
$$\begin{aligned} \langle \lambda_1 \lambda_2 \rangle_\rho^{(\alpha,\beta_n)} &:= \int_{\Lambda_1 \times \Lambda_2} \lambda_1 \lambda_2 \mathrm{tr}[\rho\{M_1^{(\alpha)}(d\lambda_1) \otimes M_2^{(\beta n)}(d\lambda_2)\}] \\ &= \mathrm{tr}[\rho(W_1^{(\alpha)} \otimes W_2^{(\beta_n)})], \quad n = 1, 2, \end{aligned} \tag{21}$$
represents the expectation value of the product $\lambda_1 \lambda_2$ of the observed outcomes.

  Here, by $W_1^{(\alpha)}$ and $W_2^{(\beta_i)}$, we denote the self-adjoint bounded linear operators on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, defined by the relations

$$\begin{aligned} W_1^{(\alpha)} &= \int_{\not\geq_1} \lambda_1 M_1^{(\alpha)}(d\lambda_1), & \left\| W_1^{(\alpha)} \right\| \leq \mathrm{C}_1 \\ W_2^{(\beta_n)} &= \int_{\not\geq_2} \lambda_2 M_2^{(\beta_n)}(d\lambda_2), & \left\| W_2^{(\beta_n)} \right\| \leq \mathrm{C}_2, \qquad n = 1, 2, \end{aligned} \tag{22}$$

  and corresponding, respectively, to the outcomes in Λ_1 (on the side of Alice) and to the outcomes in Λ_2 (on the side of Bob).

Suppose that a bipartite quantum system $\mathcal{S}_q^{(1)}+\mathcal{S}_q^{(2)}$ is initially in a separable state and, for concreteness, denote a separable state by ρ_s. Let

$$\rho_s = \sum_m \xi_m \rho_1^{(m)} \otimes \rho_2^{(m)}, \qquad \xi_m > 0, \qquad \sum_m \xi_m = 1, \tag{23}$$

be a possible separable representation of ρ_s.

For any separable representation (23) of ρ_s, we derive, using (21) and (16), the following upper bound:

$$\begin{aligned} \left| \langle \lambda_1 \lambda_2 \rangle_{\rho_s}^{(\alpha,\beta_1)} - \langle \lambda_1 \lambda_2 \rangle_{\rho_s}^{(\alpha,\beta_2)} \right| &\leq \sum_m \xi_m \mathrm{C}_1 \left| \mathrm{tr}[\rho_2^{(m)} W_2^{(\beta_1)}] - \mathrm{tr}[\rho_2^{(m)} W_2^{(\beta_2)}] \right| \\ &\leq \mathrm{C}_1 \mathrm{C}_2 - \frac{\mathrm{C}_1}{\mathrm{C}_2} \sum_m \xi_m \mathrm{tr}[\rho_2^{(m)} W_2^{(\beta_1)}] \mathrm{tr}[\rho_2^{(m)} W_2^{(\beta_2)}] \\ &= \mathrm{C}_1 \mathrm{C}_2 - \frac{\mathrm{C}_1}{\mathrm{C}_2} \langle \lambda_2 \lambda_2' \rangle_{\sigma_2}^{(\beta_1,\beta_2)}, \end{aligned} \tag{24}$$

where:
(i) σ_2 is the separable density operator

$$\sigma_2 := \sum_m \xi_m \rho_2^{(m)} \otimes \rho_2^{(m)} \tag{25}$$

on $\mathcal{H}_2 \otimes \mathcal{H}_2$, corresponding to a separable representation (23) of ρ_s;
(ii) $W_2^{(\beta_n)}$, $n = 1, 2$, are the self-adjoint bounded linear operators on $\mathcal{H}_2$, defined by (22), and the notation $\{\cdot\}_{sym}$ is introduced by (12);
(iii) $\langle \lambda_2 \lambda_2' \rangle_{\sigma_2}^{(\beta_1,\beta_2)}$ is the expectation value

$$\begin{aligned} \langle \lambda_2 \lambda_2' \rangle_{\sigma_2}^{(\beta_1,\beta_2)} &= \mathrm{tr}[\sigma_2 \{ W_2^{(\beta_1)} \otimes W_2^{(\beta_2)} \}_{sym}] \\ &= \int_{\not\geq_2 \times \Lambda_2} \lambda_2 \lambda_2' \mathrm{tr}[\sigma_2 \{ M_2^{(\beta_1)}(d\lambda_2) \otimes M_2^{(\beta_2)}(d\lambda_2') \}_{sym}], \end{aligned} \tag{26}$$

under the joint experiment, with outcomes in $\Lambda_2 \times \Lambda_2$, represented by the POV measure

$$\{ M_2^{(\beta_1)}(B_2) \otimes M_2^{(\beta_2)}(B_2') \}_{sym}, \tag{27}$$

for any subsets B_2, B_2' of Λ_2, and performed on the bipartite quantum system $\mathcal{S}_q^{(2)}+\mathcal{S}_q^{(2)}$ in the state σ_2.

The inequality (24) *establishes the relation between three joint experiments and is valid for any initial separable state.*

In general, the mean values in the left and the right hand sides of (24) *refer to joint experiments on different bipartite systems, namely, on* $\mathcal{S}_q^{(1)}+\mathcal{S}_q^{(2)}$ *and* $\mathcal{S}_q^{(2)}+\mathcal{S}_q^{(2)}$, *respectively.*

Moreover, the upper bound in (24) depends on a chosen separable representation of ρ_s.

All possible separable representations of ρ_s induce, via (25), the set

$$\mathcal{D}_{\mathcal{H}_2 \otimes \mathcal{H}_2}^{(\rho_s)} \tag{28}$$

of density operators σ_2 on $\mathcal{H}_2 \otimes \mathcal{H}_2$. For any two density operators $\sigma_2^{(1)}$ and $\sigma_2^{(2)}$ in this set and any non-negative real number $\alpha \leq 1$, there exists a separable representation of ρ_s such that the density operator (25), corresponding to this separable representation, coincides with the convex linear combination

$$\alpha \sigma_2^{(1)} + (1 - \alpha) \sigma_2^{(2)}. \tag{29}$$

Hence, the set $\mathcal{D}_{\mathcal{H}_2 \otimes \mathcal{H}_2}^{(\rho_s)}$ is convex linear.

Furthermore, since the upper bound (24) is valid for any separable representation of ρ_s, we have the following inequality:

$$\left|\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(\alpha,\beta_1)} - \langle\lambda_1\lambda_2\rangle_{\rho_s}^{(\alpha,\beta_2)}\right| \leq \inf_{\sigma_2\in\mathcal{D}_{\mathcal{H}_2\otimes\mathcal{H}_2}^{(\rho_s)}} \left\{\mathrm{C}_1\mathrm{C}_2 - \frac{\mathrm{C}_1}{\mathrm{C}_2}\langle\lambda_2\lambda_2'\rangle_{\sigma_2}^{(\beta_1,\beta_2)}\right\}. \tag{30}$$

Consider now the case where both quantum sub-systems $\mathcal{S}_q^{(1)}$ and $\mathcal{S}_q^{(2)}$ (possibly different) are described by the same Hilbert space: $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$.

In this case, if a state ρ_s admits a separable representation of the form

$$\rho_s = \sum_m \xi_m \rho^{(m)} \otimes \rho^{(m)}, \quad \xi_m > 0, \quad \sum_m \xi_m = 1, \tag{31}$$

then, for this separable representation of ρ_s, the corresponding density operator (25) coincides with ρ_s and the corresponding upper bound (24) has the form:

$$\left|\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(\alpha,\beta_1)} - \langle\lambda_1\lambda_2\rangle_{\rho_s}^{(\alpha,\beta_2)}\right| \leq \mathrm{C}_1\mathrm{C}_2 - \frac{\mathrm{C}_1}{\mathrm{C}_2}\langle\lambda_2\lambda_2'\rangle_{\rho_s}^{(\beta_1,\beta_2)} \tag{32}$$

and refers to the mean values in the same quantum state although to the joint experiments on different bipartite systems. Namely, in the left-hand side of (32) the averages correspond to the joint experiments on $\mathcal{S}_q^{(1)} + \mathcal{S}_q^{(2)}$ while in the right hand side on $\mathcal{S}_q^{(2)} + \mathcal{S}_q^{(2)}$.

Using (16), we further generalize (24) to the case of any linear combination of the expectation values:

$$\begin{aligned}
&\left|\gamma_1\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(\alpha,\beta_1)} + \gamma_2\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(\alpha,\beta_2)}\right| \\
&= \gamma_0\left|\frac{\gamma_1}{\gamma_0}\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(\alpha,\beta_1)} + \frac{\gamma_2}{\gamma_0}\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(\alpha,\beta_2)}\right| \\
&\leq \gamma_0\mathrm{C}_1\sum_m \xi_m\left|\frac{\gamma_1}{\gamma_0}\mathrm{tr}[\rho_2^{(m)}W_2^{(\beta_1)}] + \frac{\gamma_2}{\gamma_0}\mathrm{tr}[\rho_2^{(m)}W_2^{(\beta_2)}]\right| \\
&\leq \gamma_0\mathrm{C}_1\mathrm{C}_2\left\{1 + \frac{\gamma_1\gamma_2}{\gamma_0^2\mathrm{C}_2^2}\ \langle\lambda_2\lambda_2'\rangle_{\sigma_2}^{(\beta_1,\beta_2)}\right\} \\
&= \gamma_0\mathrm{C}_1\mathrm{C}_2 + \frac{\gamma_1\gamma_2}{\gamma_0}\frac{\mathrm{C}_1}{\mathrm{C}_2}\langle\lambda_2\lambda_2'\rangle_{\sigma_2}^{(\beta_1,\beta_2)},
\end{aligned} \tag{33}$$

where γ_1, γ_2 are any real numbers with $|\gamma_1| + |\gamma_2| \neq 0$ and $\gamma_0 := \max_{i=1,2}|\gamma_i|$.

3.2 Identical quantum sub-systems

Consider now the situation where quantum sub-systems are identical: $\mathcal{S}_q^{(1)} = \mathcal{S}_q^{(2)} = \mathcal{S}_q$.

In this case, $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ and, for both Boson and Fermi statistics, an initial state must satisfy the relation

$$\mathrm{S}_2\rho = \rho, \tag{34}$$

where we denote by S_2 the symmetrization operator on $\mathcal{H}\otimes\mathcal{H}$ (see [8], page 53).

Moreover, for any joint experiment on $\mathcal{S}_q + \mathcal{S}_q$ of the Alice/Bob type, with outcomes in $\Lambda_1\times\Lambda_2$, the POV measure of each individual (marginal) experiment on the side of Alice or Bob must have a symmetrized tensor product form and be specified by a set Λ_i, $i = 1, 2$, of outcomes but not by the "side" of the tensor product.

The latter means that, for this type of a joint experiment on $\mathcal{S}_q + \mathcal{S}_q$, the POV measure must have the following form:

$$M(B_1\times B_2) = \{M_1(B_1)\otimes M_2(B_2)\}_{sym} = \frac{1}{2}\{M_1(B_1)\otimes M_2(B_2) + M_2(B_2)\otimes M_1(B_1)\}, \tag{35}$$

for any subset B_1 of Λ_1 and any subset B_2 of Λ_2.

However, for further calculation of traces in a state ρ, satisfying the condition (34), the symmetrization (35) is not essential since, for this state,

$$\mathrm{tr}[\rho\{V_1 \otimes V_2\}_{sym}] = \mathrm{tr}[\rho(V_1 \otimes V_2)], \tag{36}$$

for any bounded V_1 and V_2 on $\mathcal{H}$.

Consider two joint experiments (of the Alice/Bob type) on $\mathcal{S}_q + \mathcal{S}_q$, represented by the POV measures

$$\begin{aligned} M^{(\alpha,\beta_1)}(B_1 \times B_2) &= \{M_1^{(\alpha)}(B_1) \otimes M_2^{(\beta_1)}(B_2)\}_{sym}, \\ M^{(\alpha,\beta_2)}(B_1 \times B_2) &= \{M_1^{(\alpha)}(B_1) \otimes M_2^{(\beta_2)}(B_2)\}_{sym}, \end{aligned} \tag{37}$$

and performed on a bipartite quantum system $\mathcal{S}_q$+$\mathcal{S}_q$ being initially in a state ρ, satisfying (34).

In (37), the parameters α and β (of any nature) characterize set-ups of the corresponding (marginal) experiments with outcomes in Λ_1 and in Λ_2, respectively.

For each of the joint experiments (37), the formula

$$\begin{aligned} \langle \lambda_1 \lambda_2 \rangle_{\rho}^{(\alpha,\beta_n)} &= \int \lambda_1 \lambda_2 \mathrm{tr}[\rho\{M_1^{(\alpha)}(d\lambda_1) \otimes M_2^{(\beta_n)}(d\lambda_2)\}_{sym}] \\ &= \mathrm{tr}[\rho\{W_1^{(\alpha)} \otimes W_2^{(\beta_n)}\}_{sym}], \quad n = 1, 2, \end{aligned} \tag{38}$$

represents the expectation value of the product $\lambda_1\lambda_2$ of the observed outcomes. Here, $W_1^{(\alpha)}$ and $W_2^{(\beta_n)}$ are the self-adjoint bounded linear operators on $\mathcal{H}$, defined by (22) and corresponding to the observed outcomes in Λ_1 (on the side of Alice) and in Λ_2, (on the side of Bob), respectively.

Suppose that a bipartite quantum system $\mathcal{S}_q$+$\mathcal{S}_q$ is initially in a separable quantum state ρ_s and let

$$\rho_s = \sum_m \frac{\xi_m}{2}\{\rho_m \otimes \rho'_m + \rho'_m \otimes \rho_m\} = \sum_m \xi_m \{\rho_m \otimes \rho'_m\}_{sym}, \quad \xi_m > 0, \quad \sum_m \xi_m = 1, \tag{39}$$

be a separable representation of ρ_s.

For any separable representation (39) of ρ_s, we derive, similarly to (24), the following quantum inequality:

$$\begin{aligned} &\left|\langle \lambda_1\lambda_2 \rangle_{\rho_s}^{(\alpha,\beta_1)} - \langle \lambda_1\lambda_2 \rangle_{\rho_s}^{(\alpha,\beta_2)}\right| \\ &\leq \sum_m \frac{\xi_m}{2} \mathrm{C}_1 \left\{\left|\mathrm{tr}[\rho_m W_2^{(\beta_1)}] - \mathrm{tr}[\rho_m W_2^{(\beta_2)}]\right| + \left|\mathrm{tr}[\rho'_m W_2^{(\beta_1)}] - \mathrm{tr}[\rho'_m W_2^{(\beta_2)}]\right|\right\} \\ &\leq \mathrm{C}_1\mathrm{C}_2 - \frac{\mathrm{C}_1}{\mathrm{C}_2} \sum_m \frac{\xi_m}{2} \left\{\mathrm{tr}[\rho_m W_2^{(\beta_1)}]\ \mathrm{tr}[\rho_m W_2^{(\beta_2)}] + \mathrm{tr}[\rho'_m W_2^{(\beta_1)}]\ \mathrm{tr}[\rho'_m W_2^{(\beta_2)}]\right\} \\ &\leq \mathrm{C}_1\mathrm{C}_2 - \frac{\mathrm{C}_1}{\mathrm{C}_2} \langle \lambda_2 \lambda'_2 \rangle_{\sigma}^{(\beta_1,\beta_2)}, \end{aligned} \tag{40}$$

where

$$\sigma = \sum_m \frac{\xi_m}{2}\{\rho_m \otimes \rho_m + \rho'_m \otimes \rho'_m\} \tag{41}$$

is the separable density operator on $\mathcal{H} \otimes \mathcal{H}$, corresponding to a separable representation (39) of ρ_s, and

$$\begin{aligned} \langle \lambda_2 \lambda'_2 \rangle_{\sigma}^{(\beta_1,\beta_2)} &= \mathrm{tr}[\sigma\{W_2^{(\beta_1)} \otimes W_2^{(\beta_2)}\}_{sym}] \\ &= \int \lambda_2 \lambda'_2 \mathrm{tr}[\sigma\{M_2^{(\beta_1)}(d\lambda_2) \otimes M_2^{(\beta_2)}(d\lambda'_2)\}_{sym}] \end{aligned} \tag{42}$$

is the expectation value under the joint experiment, with outcomes in $\Lambda_2 \times \Lambda_2$ (both outcomes are on the side of Bob), described by the POV measure

$$\{M_2^{(\beta_1)}(B_2) \otimes M_2^{(\beta_2)}(B_2')\}_{sym}, \tag{43}$$

for any subsets B_2, B_2' of Λ_2, and performed on $\mathcal{S}_q + \mathcal{S}_q$ in the state σ, possibly different from ρ_s.

Notice that, in the state σ :

$$\langle \lambda_2 \lambda_2' \rangle_\sigma^{(\beta_1,\beta_1)} \geq 0. \tag{44}$$

Quite similarly to our derivation of (33), for any separable state ρ_s, we have:

$$\left| \gamma_1 \langle \lambda_1 \lambda_2 \rangle_{\rho_s}^{(\alpha,\beta_1)} + \gamma_2 \langle \lambda_1 \lambda_2 \rangle_{\rho_s}^{(\alpha,\beta_2)} \right| \leq \gamma_0 \mathrm{C}_1 \mathrm{C}_2 + \frac{\gamma_1 \gamma_2}{\gamma_0} \frac{\mathrm{C}_1}{\mathrm{C}_2} \langle \lambda_2 \lambda_2' \rangle_\sigma^{(\beta_1,\beta_2)}, \tag{45}$$

where γ_1, γ_2 are any real numbers with $|\gamma_1| + |\gamma_2| \neq 0$ and $\gamma_0 := \max_{i=1,2} |\gamma_i|$.

3.2.1 Quantum analogues of Bell's inequality

Consider further the case where, in (19), $\mathrm{C}_1 = \mathrm{C}_2 = \mathrm{C}$.

Assume that the (marginal) experiments on the sides of Alice and Bob are similar - in the sense that

$$\int \lambda_2 M_2^{(\beta_1)}(d\lambda_2) = \int \lambda_1 M_1^{(\beta_1)}(d\lambda_1) \tag{46}$$
$$\Leftrightarrow \quad W_2^{(\beta_1)} = W_1^{(\beta_1)}.$$

The condition (46) *does not represent the Bell correlation restrictions* [2-4] *on the observed outcomes on the sides of Alice and Bob and is usually fulfilled*[4] *under Alice/Bob joint quantum experiments.*

Under the condition (46), the expectation value

$$\begin{aligned} \langle \lambda_2 \lambda_2' \rangle_\sigma^{(\beta_1,\beta_2)} &= \mathrm{tr}[\sigma \{W_2^{(\beta_1)} \otimes W_2^{(\beta_2)}\}_{sym}] \\ &= \mathrm{tr}[\sigma \left\{ \int \lambda_2 \lambda_2' \{M_2^{(\beta_1)}(d\lambda_2) \otimes M_2^{(\beta_2)}(d\lambda_2')\}_{sym} \right\}] \\ &= \int \lambda_1 \lambda_2' \mathrm{tr}[\sigma \{M_1^{(\beta_1)}(d\lambda_1) \otimes M_2^{(\beta_2)}(d\lambda_2')\}_{sym}] \\ &= \langle \lambda_1 \lambda_2 \rangle_\sigma^{(\beta_1,\beta_2)}, \end{aligned} \tag{47}$$

and, hence, the inequality (40) takes the Bell form:

$$\left| \langle \lambda_1 \lambda_2 \rangle_{\rho_s}^{(\alpha,\beta_1)} - \langle \lambda_1 \lambda_2 \rangle_{\rho_s}^{(\alpha,\beta_2)} \right| \leq \mathrm{C}^2 - \langle \lambda_1 \lambda_2 \rangle_\sigma^{(\beta_1,\beta_2)}, \tag{48}$$

with *all three mean values referring to Alice/Bob joint experiments.*

However, in general, $\sigma \neq \rho_s$ and this means that, under these joint experiments, the initial states of $\mathcal{S}_q + \mathcal{S}_q$ may be different.

We call (48) *a quantum analogue of the original Bell inequality for a separable quantum state.*

To different separable representations of a separable state ρ_s, there correspond different terms $\langle \lambda_1 \lambda_2 \rangle_\sigma^{(\beta_1,\beta_2)}$ in the right-hand side of (48).

In general, for a separable state, any quantum inequality (48) need not coincide with the original Bell inequality and, hence, for this state, the original Bell inequality may be violated.

Let specify the conditions under which a separable state of $\mathcal{S}_q + \mathcal{S}_q$ satisfies the original Bell inequality. We suppose that the condition (46) is fulfilled.

[4]For identical quantum sub-systems, this is a condition on identical measurement devices, used on both sides, for example, identical polarization analyzers.

1. Let an initial separable state ρ_s on $\mathcal{H} \otimes \mathcal{H}$ admit a representation of the special form

$$\rho_s = \sum_m \xi_m \rho_m \otimes \rho_m, \quad \xi_m > 0, \quad \sum_m \xi_m = 1. \tag{49}$$

Then, from (41) it follows that, for this separable representation,

$$\sigma = \rho_s \tag{50}$$

and, hence, the corresponding quantum inequality (48) reduces to

$$\left| \langle \lambda_1 \lambda_2 \rangle_{\rho_s}^{(\alpha,\beta_1)} - \langle \lambda_1 \lambda_2 \rangle_{\rho_s}^{(\alpha,\beta_2)} \right| \leq \mathrm{C}^2 - \langle \lambda_1 \lambda_2 \rangle_{\rho_s}^{(\beta_1,\beta_2)}, \tag{51}$$

so that *a separable state* (49) *satisfies the original Bell inequality in its perfect correlation form.*

For a separable state (49), the correlation function $\langle \lambda_1 \lambda_2 \rangle_{\rho_s}^{(\beta_1,\beta_1)}$ is always non-negative:

$$\langle \lambda_1 \lambda_2 \rangle_{\rho_s}^{(\beta_1,\beta_1)} \geq 0 \tag{52}$$

and may take any value in $[0, \mathrm{C}^2]$.

2. Consider further the situation where the marginal experiments and a separable state ρ_s are such that for some separable representation (39) of ρ_s :

$$\mathrm{tr}[\sigma\{W_1^{(\beta_1)} \otimes W_2^{(\beta_2)}\}_{sym}] = \pm \mathrm{tr}[\rho_s\{W_1^{(\beta_1)} \otimes W_2^{(\beta_2)}\}_{sym}], \tag{53}$$

or, equivalently,

$$\langle \lambda_1 \lambda_2 \rangle_{\sigma}^{(\beta_1,\beta_2)} = \pm \langle \lambda_1 \lambda_2 \rangle_{\rho_s}^{(\beta_1,\beta_2)}. \tag{54}$$

Notice, that, under the condition (49) in the point 1, the condition (53) is satisfied. However, we have specially separated these two cases since (49) represents a restriction only on a separable state while (53) is, in general, a restriction on the combination - a joint experiment plus a state.

Under the condition (54), the corresponding quantum analogue (48) reduces to

$$\left| \langle \lambda_1 \lambda_2 \rangle_{\rho_s}^{(\alpha,\beta_1)} - \langle \lambda_1 \lambda_2 \rangle_{\rho_s}^{(\alpha,\beta_2)} \right| \leq \mathrm{C}^2 \mp \langle \lambda_1 \lambda_2 \rangle_{\rho_s}^{(\beta_1,\beta_2)}, \tag{55}$$

and *coincides with the original Bell inequality, in its perfect correlation (sign "minus") or anti-correlation (sign "plus") forms.*

In (54), the correlation function $\langle \lambda_1 \lambda_2 \rangle_{\sigma}^{(\beta_1,\beta_1)}$ in the state σ is always non-negative (see (44) and (47)). Hence, a necessary condition for a separable state ρ_s to satisfy the perfect correlation form of the original Bell inequality constitutes:

$$\langle \lambda_1 \lambda_2 \rangle_{\rho_s}^{(\beta_1,\beta_1)} \geq 0. \tag{56}$$

The relation

$$\langle \lambda_1 \lambda_2 \rangle_{\rho_s}^{(\beta_1,\beta_1)} \leq 0 \tag{57}$$

represents a necessary condition for a separable state.ρ_s to satisfy the perfect anti-correlation form of the Bell inequality.

The condition (54) (equivalently, (53)) is, in particular, satisfied if, for all indices m,

$$\mathrm{tr}[\rho_m W_1^{(\beta_1)}] = \pm \mathrm{tr}[\rho'_m W_1^{(\beta_1)}]. \tag{58}$$

The latter relation implies that, for a separable quantum state, *Bell's correlation restrictions*[5]*:*

$$\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(\beta_1,\beta_1)} = \pm 1 \tag{59}$$

represent a particular case of the condition (58)*, and, hence, of the condition* (54)*.*

Example 1 *Consider the example of section 2. In this example,*

$$\mathrm{C} = 1, \quad \rho_s = \rho_0, \quad \alpha = \theta_a, \quad \beta_1 = \theta_b, \quad \beta_2 = \theta_c. \tag{60}$$

For any parameters θ_a*,* θ_b*,* θ_c*, we have:*

$$\begin{aligned}
\langle\lambda_1\lambda_2\rangle_{\rho_0}^{(\theta_a,\theta_b)} &= \mathrm{tr}[\rho_0\{J^{(\theta_a)}\otimes J^{(\theta_b)}\}_{sym}] = -\cos 2\theta_a \ \cos 2\theta_b, \\
\langle\lambda_1\lambda_2\rangle_{\rho_0}^{(\theta_a,\theta_c)} &= \mathrm{tr}[\rho_0\{J^{(\theta_a)}\otimes J^{(\theta_c)}\}_{sym}] = -\cos 2\theta_a \ \cos 2\theta_c, \\
\sigma &= \frac{1}{2}\{\{|\uparrow><\uparrow|\otimes|\uparrow><\uparrow| + |\downarrow><\downarrow|\otimes|\downarrow><\downarrow|\}, \\
\langle\lambda_1\lambda_2\rangle_{\sigma}^{(\theta_b,\theta_c)} &= \mathrm{tr}[\sigma\{J^{(\theta_b)}\otimes J^{(\theta_c)}\}_{sym}] = \cos 2\theta_b \ \cos 2\theta_c.
\end{aligned} \tag{61}$$

For any θ_b*,* θ_c*,*

$$\langle\lambda_1\lambda_2\rangle_{\sigma}^{(\theta_b,\theta_c)} = -\langle\lambda_1\lambda_2\rangle_{\rho_0}^{(\theta_b,\theta_c)}, \tag{62}$$

and hence, the condition (54) is satisfied. For all θ_a*,* θ_b*,* θ_c*, the quantum inequality (48) is given by*

$$\left|\langle\lambda_1\lambda_2\rangle_{\rho_0}^{(\theta_a,\theta_b)} - \langle\lambda_1\lambda_2\rangle_{\rho_0}^{(\theta_a,\theta_c)}\right| \leq 1 + \langle\lambda_1\lambda_2\rangle_{\rho_0}^{(\theta_b,\theta_c)}, \tag{63}$$

and coincides with the anti-correlation form of the original Bell inequality.

Thus, under the condition (46) *on similarity*[6] *of experimental devices on the side of Alice and the side of Bob, any separable quantum state of* $\mathcal{S}_q+\mathcal{S}_q$ *satisfies a quantum analogue of the original Bell inequality.*

Under the sufficient conditions, specified in items 1 and 2, a separable state satisfies the original Bell inequality, in its perfect correlation or anti-correlation forms.

4 Extended CHSH inequality

In this section, we introduce the extended CHSH inequality for any linear combination of mean values. Based on our results in section 3.1, we prove that this inequality is valid for any separable state.

Consider four joint experiments of the Alice/Bob type on a bipartite quantum system $\mathcal{S}_q^{(1)}+\mathcal{S}_q^{(2)}$ on $\mathcal{H}_1\otimes\mathcal{H}_2$. Let all these experiments have outcomes in $\Lambda_1\times\Lambda_2$ (see (19)) and be described by the POV measures:

$$\begin{aligned}
M^{(a,b)}(B_1\times B_2) &= M_1^{(a)}(B_1)\otimes M_2^{(b)}(B_2), \qquad & M^{(c,b)}(B_1\times B_2) &= M_1^{(c)}(B_1)\otimes M_2^{(b)}(B_2), \\
M^{(c,d)}(B_1\times B_2) &= M_1^{(c)}(B_1)\otimes M_2^{(d)}(B_2), \qquad & M^{(a,d)}(B_1\times B_2) &= M_1^{(a)}(B_1)\otimes M_2^{(d)}(B_2),
\end{aligned} \tag{64}$$

for any subset $B_1\subseteq\Lambda_1$ and any subset $B_2\subseteq\Lambda_2$.

In (64), the parameters a, b, c, d are of any nature and a, b refer to the set-ups of the experiments with outcomes in Λ_1 (the "side" of Alice) while b, d refer to the set-ups of the experiments with outcomes in Λ_2 (the "side" of Bob).

[5]Introduced for the derivation of the original Bell inequality in the frame of a LHV model (see [4], sections 2, 4).

[6]We would like to underline once more that this is not a condition on any correlation between the observed outcomes on the sides of Alice and Bob.

Suppose that all four joint experiments (64) are performed on a bipartite quantum system $\mathcal{S}_q^{(1)}+\mathcal{S}_q^{(2)}$ in the same separable state ρ_s.

For any real numbers γ_i, $i=1,...,4$, with

$$|\gamma_1|+|\gamma_2|+|\gamma_3|+|\gamma_4|\neq 0, \tag{65}$$

let estimate the linear combination

$$\left|\gamma_1\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(a,b)}+\gamma_2\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(c,b)}+\gamma_3\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(c,d)}+\gamma_4\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(a,d)}\right| \tag{66}$$

of the mean values under four joint experiments (64).

Similarly to the derivation of (33), to any separable representation (23) of ρ_s, we have:

$$\begin{aligned}
&\left|\gamma_1\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(a,b)}+\gamma_2\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(c,b)}+\gamma_3\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(c,d)}+\gamma_4\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(a,d)}\right| \\
&\leq \left|\gamma_1\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(a,b)}+\gamma_4\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(a,d)}\right|+\left|\gamma_2\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(c,b)}+\gamma_3\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(c,d)}\right| \\
&=\widetilde{\gamma}_0\left|\frac{\gamma_1}{\widetilde{\gamma}_0}\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(a,b)}+\frac{\gamma_4}{\widetilde{\gamma}_0}\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(a,d)}\right|+\widetilde{\gamma}_0\left|\frac{\gamma_2}{\widetilde{\gamma}_0}\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(c,b)}+\frac{\gamma_3}{\widetilde{\gamma}_0}\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(c,d)}\right| \\
&\leq 2\widetilde{\gamma}_0\mathrm{C}_1\mathrm{C}_2+\frac{C_1}{\widetilde{\gamma}_0 C_2}\left\{\gamma_1\gamma_4+\gamma_2\gamma_3\right\}\langle\lambda_2\lambda_2'\rangle_{\sigma_2}^{(b,d)},
\end{aligned} \tag{67}$$

where $\widetilde{\gamma}_0:=\max_{i=1,...,4}|\gamma_i|$ and

$$\langle\lambda_2\lambda_2'\rangle_{\sigma_2}^{(b,d)}:=\mathrm{tr}[\sigma_2\{W_2^{(b)}\otimes W_2^{(d)}\}_{sym}],$$

with the density operator σ_2 on $\mathcal{H}_2\otimes\mathcal{H}_2$, defined by (25).

However, if, in the second line of (67), we combine the terms in another way, we derive:

$$\begin{aligned}
&\left|\ \gamma_1\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(a,b)}+\gamma_2\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(c,b)}+\gamma_3\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(c,d)}+\gamma_4\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(a,d)}\ \right| \\
&\leq\left|\ \gamma_1\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(a,b)}+\gamma_2\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(c,b)}\ \right|+\left|\ \gamma_3\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(c,d)}+\gamma_4\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(a,d)}\ \right| \\
&\leq 2\widetilde{\gamma}_0\mathrm{C}_1\mathrm{C}_2+\frac{C_1}{\widetilde{\gamma}_0 C_2}\left\{\gamma_1\gamma_2+\gamma_3\gamma_4\right\}\langle\lambda_1\lambda_1'\rangle_{\sigma_1}^{(a,c)},
\end{aligned} \tag{68}$$

where

$$\langle\lambda_1\lambda_1'\rangle_{\sigma_1}^{(a,c)}=\mathrm{tr}[\sigma_1\{W_1^{(a)}\otimes W_1^{(c)}\}_{sym}],$$

with

$$\sigma_1=\sum_m \xi_m\rho_1^{(m)}\otimes\rho_1^{(m)} \tag{69}$$

being the density operator on $\mathcal{H}_1\otimes\mathcal{H}_1$, corresponding to a separable representation (23).

From (67) and (68) it follows that, for any separable state ρ_s, the inequality

$$\left|\gamma_1\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(a,b)}+\gamma_2\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(c,b)}+\gamma_3\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(c,d)}+\gamma_4\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(a,d)}\right|\leq 2\widetilde{\gamma}_0\mathrm{C}_1\mathrm{C}_2. \tag{70}$$

holds for all real numbers γ_1, γ_2, γ_3, γ_4, with $|\gamma_1|+|\gamma_2|+|\gamma_3|+|\gamma_4|\neq 0$, such that

$$\gamma_1\gamma_4=-\gamma_2\gamma_3\quad\text{or}\quad\gamma_1\gamma_2=-\gamma_3\gamma_4. \tag{71}$$

We refer to (70) *as the extended CHSH inequality.* The original CHSH inequality

$$\left|\ \langle\lambda_1\lambda_2\rangle_{\rho_s}^{(a,b)}+\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(c,b)}+\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(c,d)}-\langle\lambda_1\lambda_2\rangle_{\rho_s}^{(a,d)}\ \right|\leq 2 \tag{72}$$

is the special case of (70) for

$$\gamma_1 = \gamma_2 = \gamma_3 = -\gamma_4, \qquad \mathrm{C}_1\mathrm{C}_2 = 1. \tag{73}$$

Similarly to our presentation in this section, it is easy to verify that the extended CHSH inequality is valid for all joint classical measurements[7].

Thus, in contrast to the situation with the original Bell inequality (A6)*, such a classical probabilistic constraint as the extended CHSH inequality is valid for any separable quantum state. Moreover, this inequality holds for a variety of linear combinations of the mean values.*

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5 Appendix: Classical measurements

Let a system $\mathcal{S}$ be described in terms of some parameters $\theta \in \Theta$ (hidden or real) and a probability distribution π of these parameters.

An experiment, representing a non-perturbing measurement of some property A of $\mathcal{S}$, which existed before an experiment, is described by a measurable function[8] f_A on Θ, with values that are outcomes under this experiment. This type of an experiment is called (see in [9, 10]) a classical measurement.

Under this type of *ideal* experiment on $\mathcal{S}$, the probability distribution of outcomes is an image of an initial probability distribution π and does not depend on an arrangement of a measurement. Notice that Bell's LHV model ([4], sections 2, 4) describes a perturbing classical experiment, where the probabilities of the system properties, which existed before an experiment, are modified by this measurement.

Any joint classical measurement on two system properties (say A and D) is described by two real-valued functions f_A, f_D on Θ, with values equal to the outcomes λ_1 and λ_2, observed under this joint classical measurement. The expectation value $\langle\lambda_1\lambda_2\rangle_{cl}^{(A\&D)}$ of the product of the observed outcomes is given by

$$\langle\lambda_1\lambda_2\rangle_{cl}^{(A\&D)} = \int_\Theta f_A(\theta) f_D(\theta)\pi(d\theta). \tag{A1}$$

Notice that, for the probabilistic description of any joint experiment, classical or quantum, it is not essential whether or not individual (i.e. marginal) experiments are separated in time or space.

Suppose now that we have two joint classical measurements of properties $A\&D_1$ and $A\&D_2$ and

$$|f_A(\theta)| \le \mathrm{C}_1, \qquad |f_{D_1}(\theta)| \le \mathrm{C}_2, \qquad |f_{D_2}(\theta)| \le \mathrm{C}_2, \tag{A2}$$

for all those $\theta \in \Theta$ where π does not vanish.

Consider, in a very general setting, the relation between the expectation values

$$\langle\lambda_1\lambda_2\rangle_{cl}^{(A\&D_1)} = \int_\Theta f_A(\theta) f_{D_1}(\theta)\pi(d\theta), \qquad \langle\lambda_1\lambda_2\rangle_{cl}^{(A\&D_2)} = \int_\Theta f_A(\theta)\ f_{D_2}(\theta)\pi(d\theta), \tag{A3}$$

$$\langle\lambda_1\lambda_2\rangle_{cl}^{(D_1\&D_2)} = \int_\Theta f_{D_1}(\theta) f_{D_2}(\theta)\pi(d\theta),$$

under three joint classical measurements on properties

$$A\&D_1, \qquad A\&D_2, \qquad D_1\&D_2 \tag{A4}$$

[7] For the description of a classical measurement, see appendix.

[8] In classical probability, these functions are called random variables.

of $\mathcal{S}$. Due to the inequality (16), we have:

$$\begin{aligned}
&\left|\langle\lambda_1\lambda_2\rangle_{cl}^{(A\&D_1)} - \langle\lambda_1\lambda_2\rangle_{cl}^{(A\&D_2)}\right| \qquad \text{(A5)}\\
&\leq \int_\Theta |f_A(\theta)| \;\mid f_{D_1}(\theta) - f_{D_2}(\theta)|\,\pi(d\theta)\\
&\leq \mathrm{C}_1 \int_\Theta \mid f_{D_1}(\theta) - f_{D_2}(\theta)\mid \pi(d\theta)\\
&\leq \mathrm{C}_1\mathrm{C}_2 - \frac{\mathrm{C}_1}{\mathrm{C}_2}\langle\lambda_1\lambda_2\rangle_{cl}^{(D_1\&D_2)}.
\end{aligned}$$

If $\mathrm{C}_1 = \mathrm{C}_2 = 1$ then (A5) coincides in form with the original Bell inequality [2-4] for the case of the perfect correlation of the observed outcomes.

In order to distinguish (A5) from numerous generalizations and strengthenings of Bell's inequality, in this paper, for any joint experiments, classical or quantum, we refer to an inequality[9]

$$\left|\langle\lambda_1\lambda_2\rangle^{(A\&D_1)} - \langle\lambda_1\lambda_2\rangle^{(A\&D_2)}\right| \leq \mathrm{C}_1\mathrm{C}_2 - \frac{\mathrm{C}_1}{\mathrm{C}_2}\langle\lambda_1\lambda_2\rangle^{(D_1\&D_2)} \qquad \text{(A6)}$$

as the *original Bell inequality, in its perfect correlation form.*

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[9] See [5], for a sufficient condition on the derivation of this inequality under joint experiments on a system of any type.