Exercises and Solutions to the Concentrated Advanced Course on Stochastic Partial Differential Equations

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1 Introduction

In September 1998 professor Helge Holden gave an advanced course on stochastic partial differential equations at the Centre for Mathematical Physics and Stochastics (MaPhySto). The course consisted of lectures and tutorial classes.

This note contains the exercises with suggested solutions discussed in the classes. Notation and necessary theoretical background to understand the exercises can be found in the monograph, [HØUZ]. All references to exercises, equations, theorems etc below are from [HØUZ].

We have included an annotated list of references which is recommended for the ones interested in White Noise Analysis and its applications.

2 Exercises

Day 1

Exercise 1. *Exercise 2.6*

Exercise 2. *Consider* $w(\phi,\omega) = \langle\omega,\phi\rangle$, $\phi \in L^2(\mathbb{R}^d), \omega \in \mathcal{S}'(\mathbb{R}^d)$. *Show that* $w(\phi)$ *is normally distributed with expectation zero and variance* $\|\phi\|^2$.

Exercise 3. *Consider* $W_\phi(x,\omega) = w(\phi_x,\omega)$, *for* $\phi \in L^2(\mathbb{R}^d), x \in \mathbb{R}^d$, *where* $\phi_x(y) := \phi(y-x)$. *Prove that* $W_\phi(x,\omega)$ *has properties (2.1.20)-(2.1.22).*

Exercise 4. *Show that* $E[H_\alpha] = 0$ *for* $\alpha \neq 0$, *where* H_α *is defined in Def. 2.2.1-2, pp.19-20.*

Exercise 5. *Let* $f \in L^2(\mu_m)$ *with chaos expansion* $f = \sum_{\alpha\in\mathcal{J}} c_\alpha H_\alpha$. *Find* $E[f]$.

Exercise 6. *Consider the* 1*-dimensional,* 1*-parameter smoothed white noise*

$$w(\phi,\omega) = \sum_{j=1}^{\infty}(\phi,\xi_j)H_{\epsilon_j}(\omega)$$

where ξ_j *is the j'th Hermite function. Find the variance of* $w(\phi)$ *by using its chaos expansion.*

Exercise 7. *Let $\{\xi_j\}_{j\in\mathbb{N}}$ be the Hermite functions and define $T=\sum_{j=1}^{\infty}\xi_j(t)\xi_j(\cdot)$. Show, by using Th. 2.3.1, p. 28, that $T\in\mathcal{S}'(\mathbb{R}^d)$. Moreover, prove that $T=\delta_t$ in the sense of distributions, where δ_t is the Dirac δ-function at t.*

Exercise 8. *Let $N=m=d=1$. Find the (formal) chaos expansion of $W_t(\omega)=\langle\omega,\delta_t\rangle$. Show that $W_t\in(\mathcal{S})_{-\rho}$ for all $\rho\in[0,1]$.*

Exercise 9. *Show that $W_\phi(x,\omega)\in(\mathcal{S})_1$ for $\phi\in\mathcal{S}(\mathbb{R})$. (Let for simplicity $N=m=d=1$).*

Day 2

Exercise 1. *Exercises 2.9 and 2.8 a-c*

Exercise 2. *Show that $w(\phi)\diamond w(\psi)=w(\phi)\cdot w(\psi)$ if $\operatorname{supp}\phi\cap\operatorname{supp}\psi=\emptyset$.*

Exercise 3. *Show that*

$$w^{\diamond 2}(\phi)\diamond w(\psi)=w^2(\phi)w(\psi)-\|\phi\|^2w(\psi)-2(\phi,\psi)w(\phi)\tag{2.1}$$

Moreover, if $\operatorname{supp}\phi\cap\operatorname{supp}\psi=\emptyset$ then

$$w^{\diamond 2}(\phi)\diamond w(\psi)=w^{\diamond 2}(\phi)w(\psi)\tag{2.2}$$

Exercise 4. *Variation of exercise 2.24: Use Th. 2.5.9, p. 52, to find the Skorohod integrals in exercise 2.24 a-c.*

Day 3

Exercise 1. *Exercises 2.8 e+f, 2.11, 2.13, 2.15 and 2.24 d*

Exercise 2. *Exercises 3.4 a-c and 3.5 a+d (In exercise a, use $X(0)=G$ instead of $X(T)=G$).*

Day 4

Exercise 1. *Exercise 4.4*

Exercise 2. *Use Cor. 4.3.2, p. 150, to show that the solution of*

$$\begin{cases}\frac{\partial U}{\partial t}=\frac{1}{2}\Delta U+U\diamond W_t, & (t,x)\in\mathbb{R}_+\times\mathbb{R}^d\\ U(0,x)=f(x), & x\in\mathbb{R}^d\end{cases}$$

is

$$U(t,x)=\hat{E}\left[f(b(t))\right]\diamond\exp(B(t)-\frac{1}{2}t)$$

Exercise 3. *Assume that $f(x)\in(\mathcal{S})_{-1}$ for every x and that f satisfies all the conditions in Th. 4.3.1, p. 147. Prove that*

$$U(t,x)=u(t,x)\diamond X_t$$

is the solution of

$$\begin{cases}\frac{\partial U}{\partial t}=\frac{1}{2}\Delta U+U\diamond W_t, & (t,x)\in\mathbb{R}_+\times\mathbb{R}^d\\ U(0,x)=f(x), & x\in\mathbb{R}^d\end{cases}$$

where $u(t,x)$ solves

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\triangle u, & (t,x) \in I\!R_+ \times I\!R^d \\ u(0,x) = f(x), & x \in I\!R^d \end{cases}$$

and

$$dX_t = X_t\, dB_t, X_0 = 1$$

Day 5

Exercise 4. *Exercises 4.5 and 4.8*

3 Solutions to exercises

Day 1

Exercise 1:

Let $\phi_n \to \phi$ in $L^2(I\!R^d)$ when $n \to \infty$, where $\{\phi_n\}_{n\in I\!N}$ is a sequence in $\mathcal{S}(I\!R^d)$. Then $\{\langle\omega, \phi_n\rangle\}_{n\in I\!N}$ is a Cauchy sequence in $L^2(\mu)$:

$$\begin{aligned} \mathrm{E}\left[(\langle\omega, \phi_n\rangle - \langle\omega, \phi_m\rangle)^2\right] &= \mathrm{E}\left[\langle\omega, \phi_n - \phi_m\rangle^2\right] \\ &= ||\phi_n - \phi_m||^2 \to 0 \end{aligned}$$

whenever $n, m \to \infty$. Hence, $\{\langle\omega, \phi_n\rangle\}_{n\in I\!N}$ is convergent in $L^2(\mu)$ and we denote its limit $\langle\omega, \phi\rangle$. It is easily shown that

$$\mathrm{E}\left[\langle\omega, \phi\rangle^2\right] = ||\phi||^2$$

and

$$\mathrm{E}\left[\langle\omega, \phi\rangle\langle\omega, \psi\rangle\right] = (\phi, \psi)$$

for any $\psi \in \mathcal{S}(I\!R^d)$.

Let $\{\psi_n\}_{n\in I\!N}$ be another sequence in $\mathcal{S}(I\!R^d)$ converging to ϕ in $L^2(I\!R^d)$. Then

$$\mathrm{E}\left[(\langle\omega, \psi_n\rangle - \langle\omega, \phi\rangle)^2\right] = ||\psi_n||^2 - 2(\psi_n, \phi) + ||\phi||^2$$

But $||\psi_n|| \to ||\phi||$ and $(\psi_n, \phi) \to ||\phi||^2$ when $n \to \infty$. Therefore, $\langle\omega, \psi_n\rangle$ converges to $\langle\omega, \phi\rangle$. The definition of $\langle\omega, \phi\rangle$ is thus independent of the choice of the approximating sequence.

Exercise 2:

From the Bochner-Minlos Theorem, p. 12, the characteristic function of $\langle\omega, \phi\rangle$ is

$$\mathrm{E}\left[\exp(i\langle\omega, \phi\rangle)\right] = \exp(-\frac{1}{2}||\phi||^2)$$

Thus, $\langle\omega, \phi\rangle$ is normally distributed with zero expectation and variance $||\phi||^2$.

Exercise 3:

Property (2.1.22): Since

$$||\phi_x||^2 = \int_{\mathbb{R}^d} \phi^2(y-x)\,dy = ||\phi||^2$$

we get from exercise 2 above that $W_\phi(x) \sim \mathcal{N}\left(0, ||\phi||^2\right)$

Property (2.1.21): We consider the case $(W_\phi(x), W_\phi(y))$ for $x, y \in \mathbb{R}^d$. The general case will follow similarly. Our argument will in fact prove that $(W_\phi(x), W_\phi(y))$ is a Gaussian vector. Let $c_1, c_2 \in \mathbb{R}$ and $h > 0$. By the Bochner-Minlos Theorem on p. 12,

$$\begin{aligned} \mathrm{E}\left[\exp(ic_1 W_\phi(x+h) + ic_2 W_\phi(y+h)\right] &= \mathrm{E}\left[\exp(i\langle \omega, c_1\phi_{x+h} + c_2\phi_{y+h}\rangle)\right] \\ &= \exp(-\frac{1}{2}||c_1\phi_{x+h} + c_2\phi_{y+h}||^2) \end{aligned}$$

But

$$\begin{aligned} ||c_1\phi_{x+h} + c_2\phi_{y+h}||^2 &= c_1^2||\phi_{x+h}||^2 + 2c_1c_2(\phi_{x+h}, \phi_{y+h}) + c_2^2||\phi_{y+h}||^2 \\ &= c_1^2||\phi||^2 + 2c_1c_2(\phi_x, \phi_y) + c_2^2||\phi||^2 \\ &= [c_1,\, c_2]\begin{bmatrix} ||\phi||^2 & (\phi_x, \phi_y) \\ (\phi_x, \phi_y) & ||\phi||^2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \end{aligned}$$

Hence, $(W_\phi(x+h), W_\phi(y+h))$ is Gaussian with zero expectation and covariance matrix

$$\begin{bmatrix} ||\phi||^2 & (\phi_x, \phi_y) \\ (\phi_x, \phi_y) & ||\phi||^2 \end{bmatrix}$$

which we see is independent of h. We have proved stationarity (and Gaussianity).

Property (2.1.20): If $\mathrm{supp}\phi_{x_1} \cap \mathrm{supp}\phi_{x_2} = \emptyset$, then

$$\mathrm{E}\left[W_\phi(x_1)W_\phi(x_2)\right] = (\phi_{x_1}, \phi_{x_2}) = 0$$

Hence, since $(W_\phi(x_1), W_\phi(x_2))$ is Gaussian and $W_\phi(x_1)$ and $W_\phi(x_2)$ are independent.

Exercise 4:

From p. 22 (proof of Th. 2.2.3),

$$\mathrm{E}\left[H_\alpha\right] = \mathrm{E}\left[H_\alpha H_0\right] = \begin{cases} 1, & \alpha = 0 \\ 0, & \alpha \neq 0 \end{cases}$$

Exercise 5:

From the exercise above,

$$\mathrm{E}\left[f\right] = \sum_\alpha c_\alpha \mathrm{E}\left[H_\alpha\right] = c_0$$

Exercise 6:

By Th. 2.2.4, p. 23:

$$\begin{aligned} \mathrm{E}\left[w^2(\phi)\right] &= \sum_{j=1}^{\infty} \epsilon_j!(\phi, \xi_j)^2 \\ &= \sum_{j=1}^{\infty} (\phi, \xi_j)^2 \\ &= ||\phi||^2 \end{aligned}$$

where we used Parseval's identity in the last equality.

Exercise 7:

Note that with $d = 1$ in Th. 2.3.1, p. 28, we have the multi-index ordering

$$\{\delta^{(j)}\}_{j=1}^{\infty} = \{(1), (2), (3), \dots\} = \{j\}_{j=1}^{\infty}$$

Recall (2.2.5), p. 19:

$$\sup_{x \in \mathbb{R}} |\xi_j(s)| = O(j^{-1/12})$$

We must find a $\theta > 0$ such that (2.4.6), p. 29, holds, i.e.

$$\begin{aligned} \sum_{j=1}^{\infty} \xi_j^2(t)(\delta^{(j)})^{-\theta} &\leq K \sum_{j=1}^{\infty} j^{-1/6} \cdot j^{-\theta} \\ &= K \sum_{j=1}^{\infty} j^{-(1/6+\theta)} \end{aligned}$$

This sum is finite if $\theta > 5/6$. Hence, $T \in \mathcal{S}'(\mathbb{R})$.

Let $\phi \in \mathcal{S}(\mathbb{R})$. By definition of δ_t:

$$\langle \delta_t, \phi \rangle = \phi(t)$$

Write $\phi = \sum_{j=1}^{\infty} (\phi, \xi_j)\xi_j$. Then

$$\begin{aligned} \langle T, \phi \rangle &= \sum_{j=1}^{\infty} \xi_j(t)(\phi, \xi_j) \\ &= \sum_{j=1}^{\infty} (\phi, \xi_j)\xi_j(t) \\ &= \phi(t) \end{aligned}$$

Hence, $T = \delta_t$.

Exercise 8:

From exercise 7 above, $\delta_t = \sum_{j=1}^{\infty} \xi_j(t)\xi_j$. We calculate (formally)

$$\begin{aligned} \langle \omega, \delta_t \rangle &= \sum_{j=1}^{\infty} \xi_j(t)\langle \omega, \xi_j \rangle \\ &= \sum_{j=1}^{\infty} \xi_j(t) H_{\epsilon_j}(\omega) \end{aligned}$$

Note that $\langle \omega, \delta_t \rangle$ is *only* a suggestive notation (we are not allowed to write the dual pairing of two distributions). But its formal chaos expansion will define an element in $(\mathcal{S})_{-\rho}$, as we now prove:

From Def. 2.3.2, pp. 29–30, we must find a $q > 0$ for every $\rho \in [0,1]$ such that

$$\sum_{\alpha} b_\alpha^2 (\alpha!)^{1-\rho} (2\mathbb{N})^{-\alpha q} < \infty$$

We have $b_\alpha = \xi_j(t)$ when $\alpha = \epsilon_j$, and zero otherwise. When $\alpha = \epsilon_j$, $-q\epsilon_j$ is a vector with zeros everywhere except at coordinate j, where it has value $-q$. Formula (2.3.8), p. 29, becomes

$$(2\mathbb{N})^{-q\epsilon_j} = \Pi_i (2i)^{(-q\epsilon_j)_i} = (2j)^{-q}$$

where we have used the notation $(q\alpha)_i$ to indicate the i'th coordinate. Hence,

$$\sum_{j=1}^{\infty} \xi_j^2(t)(1!)^{1-\rho}(2j)^{-q} = \sum_{j=1}^{\infty} \xi_j^2(t)(2j)^{-q}$$
$$\leq K 2^{-q} \sum_{j=1}^{\infty} j^{-1/6} j^{-q}$$

This sum is finite whenever $q > 5/6$, and we can conclude that $W_t \in (\mathcal{S})_{-\rho}$.

Note that formally we have

$$\frac{d}{dt} B(t) = \sum_{j=1}^{\infty} \frac{d}{dt} \int_0^t \xi_j(s)\, ds H_{\epsilon_j} = W_t$$

Exercise 9:

The chaos expansion of $W_\phi(x)$ is

$$W_\phi(x) = \sum_{j=1}^{\infty} (\phi_x, \xi_j) H_{\epsilon_j}$$

We must show ((2.3.9), p. 29)

$$\sum_{j=1}^{\infty} (\phi_x, \xi_j)^2 (\epsilon_j!)^2 (2\mathbb{N})^{k\epsilon_j} < \infty$$

for all $k \in \mathbb{N}$. But

$$\sum_{j=1}^{\infty} (\phi_x, \xi_j)^2 (\epsilon_j!)^2 (2\mathbb{N})^{k\epsilon_j} = 2^k \sum_{j=1}^{\infty} (\phi_x, \xi_j)^2 j^k$$

Recall that $\phi_x \in \mathcal{S}(\mathbb{R})$, and by Th. 2.3.1 on p. 28,

$$\sum_{j=1}^{\infty} (\phi_x, \xi_j)^2 j^q < \infty$$

for all $q \geq 0$. Hence, we can conclude that $W_\phi(x) \in (\mathcal{S})_1$. Note that this will also hold for $d > 1$.

Day 2

Exercise 1:

Answer to 2.9: Let m=N=1. The chaos expansions are

$$w(\phi) = \sum_{j=1}^{\infty} (\phi, \eta_j) H_{\epsilon_j}$$
$$w(\psi) = \sum_{j=1}^{\infty} (\psi, \eta_j) H_{\epsilon_j}$$

From Def. 2.4.1, p.39,

$$w(\phi) \diamond w(\psi) = \sum_{j,k=1}^{\infty} (\phi, \eta_j) \cdot (\psi, \eta_k) H_{\epsilon_j + \epsilon_k}$$

Def. 2.2.1, p. 19, says:

$$j \neq k: \; H_{\epsilon_j+\epsilon_k} = \langle \omega, \eta_j \rangle \cdot \langle \omega, \eta_k \rangle = H_{\epsilon_j} H_{\epsilon_k}$$
$$j = k: \; H_{2\epsilon_k} = h_2(\langle \omega, \eta_j \rangle) = \langle \omega, \eta_j \rangle^2 - 1 = H_{\epsilon_j}^2 - 1$$

Thus

$$\begin{aligned} w(\phi) \diamond w(\psi) &= \sum_{j,k}^{\infty} (\phi, \eta_j)(\psi, \eta_k) H_{\epsilon_j} H_{\epsilon_k} - \sum_{j,k} (\phi, \eta_j)(\psi, \eta_k) \\ &= w(\phi) \cdot w(\psi) - \sum_{j=1}^{\infty} (\phi, \eta_j)(\psi, \eta_k) \end{aligned}$$

But if $\phi = \sum_j (\phi, \eta_j) \eta_j, \psi = \sum_k (\psi, \eta_k) \eta_k$, then

$$(\phi, \psi) = \sum_{j=1}^{\infty} (\phi, \eta_j)(\psi, \eta_j)$$

Thus, we have proved

$$w(\phi) \diamond w(\psi) = w(\phi) \cdot w(\psi) - (\phi, \eta)$$

Answer to 2.8 a: Follows by the exercise 2.9 above with $\psi = \phi$.
Answer to 2.8 b: We find the chaos expansion for $f(\omega) = B^{\diamond 2}(x, \omega)$. By (2.2.24), p. 24,

$$B(x, \omega) = \sum_{j=1}^{\infty} \int_0^x \eta_j(u)\, du \cdot H_{\epsilon_j}$$

By def. 2.4.1, p. 39,

$$\begin{aligned}
B^{\diamond 2}(x,\omega) &= \sum_{j,k=1}^{\infty} \int_0^x \eta_j(u)\,du \cdot \int_0^x \eta_k(u)\,du H_{\epsilon_j+\epsilon_k} \\
&= \sum_{j,k=1}^{\infty} \int_0^x \eta_j(u)\,du \cdot \int_0^x \eta_k(u)\,du H_{\epsilon_j} H_{\epsilon_k} \\
&\quad - \sum_{j=1}^{\infty} \left(\int_0^x \eta_j(u)\,du \right)^2 \\
&= B^2(x,\omega) - x_1 x_2 \cdots x_d
\end{aligned}$$

Answer to 2.8 c: From b above,

$$\begin{aligned}
B^2(x,\omega) &= B^{\diamond 2}(x,\omega) + x_1 x_2 \cdots x_d \\
&= x_1 x_2 \cdots x_d + \sum_{j,k=1}^{\infty} \int_0^x \eta_j(u)\,du \cdot \int_0^x \eta_k(u)\,du\, H_{\epsilon_{jk}}
\end{aligned}$$

where $\epsilon_{jk} = \epsilon_j + \epsilon_k$.

Exercise 2:

From exercise 1 (2.9) above:

$$w(\phi) \diamond w(\psi) = w(\phi) \cdot w(\psi) - (\phi, \psi)$$

Note that $(\phi, \psi) = 0$ when $\operatorname{supp} \phi \cap \operatorname{supp} \psi = \emptyset$. Hence,

$$w(\phi) \diamond w(\psi) = w(\phi) \cdot w(\psi)$$

See (2.4.3), p. 40, for a general statement.

Exercise 3:

Assume first that

$$w^{\diamond 2}(\phi) \diamond w(\psi) = w^2(\phi) w(\psi) - ||\phi||^2 w(\psi) - 2(\phi, \psi) w(\phi)$$

holds. If $\operatorname{supp} \phi \cap \operatorname{supp} \psi = \emptyset$, then

$$\begin{aligned}
w^{\diamond 2}(\phi) \diamond w(\psi) &= w^2(\phi) w(\psi) - ||\phi||^2 w(\psi) \\
&= \left(w^2(\phi) - ||\phi||^2 \right) w(\psi) \\
&= w^{\diamond 2}(\phi) w(\psi)
\end{aligned}$$

Thus, we have shown the second part of the exercise. Now, let's prove the first statement: From the definition of Wick products,

$$\begin{aligned}
w^{\diamond 2}(\phi) \diamond w(\psi) &= \left(\sum_{i,j=1}^{\infty} (\phi, \xi_i)(\phi, \xi_j) H_{\epsilon_{ij}} \right) \diamond \left(\sum_{k=1}^{\infty} (\psi, \xi_k) H_{\epsilon_k} \right) \\
&= \sum_{i,j,k} (\phi, \xi_i)(\phi, \xi_j)(\phi, \xi_k) H_{\epsilon_{ijk}}
\end{aligned}$$

Def. 2.2.1, p. 19, yields,

$$H_{\epsilon_{ijk}} = \begin{cases} \langle \omega, \xi_i \rangle \langle \omega, \xi_j \rangle \langle \omega, \xi_k \rangle, & i \neq j \neq k \\ (\langle \omega, \xi_i \rangle^2 - 1) \langle \omega, \xi_k \rangle, & i = j \neq k \\ \langle \omega, \xi_i \rangle (\langle \omega, \xi_j \rangle^2 - 1), & i \neq j = k \\ (\langle \omega, \xi_i \rangle^2 - 1) \langle \omega, \xi_j \rangle, & i = k \neq j \\ \langle \omega, \xi_i \rangle^3 - 3 \langle \omega, \xi_i \rangle, & i = j = k \end{cases}$$

Thus,

$$\begin{aligned} w^{\diamond 2}(\phi) \diamond w(\psi) = & \sum_{i \neq j \neq k} (\phi, \xi_i)(\phi, \xi_j)(\psi, \xi_k) H_{\epsilon_i} H_{\epsilon_j} H_{\epsilon_k} + \sum_{i \neq k} (\phi, \xi_i)^2 (\psi, \xi_k) H_{\epsilon_i}^2 H_{\epsilon_k} \\ & + 2 \sum_{i \neq j} (\phi, \xi_i)(\phi, \xi_j)(\psi, \xi_j) H_{\epsilon_i} H_{\epsilon_j}^2 + \sum_i (\phi, \xi_i)^2 (\psi, \xi_i) H_{\epsilon_i}^3 \\ & - \sum_{i \neq k} (\phi, \xi_i)^2 (\psi, \xi_k) H_{\epsilon_k} - 2 \sum_{i \neq j} (\phi, \xi_i)(\phi, \xi_j)(\psi, \xi_j) H_{\epsilon_i} \\ & - 3 \sum_i (\phi, \xi_i)^2 (\psi, \xi_i) H_{\epsilon_i} \\ = & \langle \omega, \phi \rangle^2 \langle \omega, \psi \rangle - \sum_{i \neq j} (\phi, \xi_i)^2 (\psi, \xi_j) H_{\epsilon_j} - 2 \sum_{i \neq j} (\phi, \xi_i)(\phi, \xi_j)(\psi, \xi_j) H_{\epsilon_i} \\ & - 3 \sum_i (\phi, \xi_i)^2 (\psi, \xi_i) H_{\epsilon_i} \end{aligned}$$

We now investigate the second and the third term:

$$\begin{aligned} \sum_{j=1}^{\infty} (\psi, \xi_j) H_{\epsilon_j} \sum_{i=1, i \neq j}^{\infty} (\phi, \xi_i)^2 &= \sum_{j=1}^{\infty} (\psi, \xi_j) H_{\epsilon_j} ||\phi||^2 - \sum_{j=1}^{\infty} (\psi, \xi_j)(\phi, \xi_j)^2 H_{\epsilon_j} \\ &= ||\phi||^2 w(\psi) - \sum_{i=1}^{\infty} (\phi, \xi_i)^2 (\psi, \xi_i) H_{\epsilon_i} \end{aligned}$$

The third term is:

$$\begin{aligned} \sum_{i=1}^{\infty} (\phi, \xi_i) H_{\epsilon_i} \sum_{j=1, j \neq i}^{\infty} (\phi, \xi_j)(\psi, \xi_j) &= \sum_{i=1}^{\infty} (\phi, \xi_i) H_{\epsilon_i} (\phi, \psi) - \sum_{i=1}^{\infty} (\phi, \xi_i)^2 (\psi, \xi_i) H_{\epsilon_i} \\ &= (\phi, \psi) w(\phi) - \sum_{i=1}^{\infty} (\phi, \xi_i)^2 (\psi, \xi_i) H_{\epsilon_i} \end{aligned}$$

Putting all together yields,

$$\begin{aligned} w^{\diamond 2}(\phi) \diamond w(\psi) = & w^2(\phi) w(\psi) - ||\phi||^2 w(\psi) + \sum_i (\phi, \xi_i)^2 (\psi, \xi_i) H_{\epsilon_i} - 2(\phi, \psi) w(\phi) \\ & + 2 \sum_i (\phi, \xi_i)^2 (\psi, \xi_i) H_{\epsilon_i} - 3 \sum_i (\phi, \xi_i)^2 (\psi, \xi_i) H_{\epsilon_i} \\ = & w^2(\phi) w(\psi) - ||\phi||^2 w(\psi) - 2(\phi, \psi) w(\phi) \end{aligned}$$

Exercise 4:

Answer to a: We apply Th. 2.5.9, p. 52, to get

$$\begin{aligned}
\int_0^T B(t_0)\delta B(t) &= \int_0^T B(t_0) \diamond W(t)\,dt \\
&= \int_0^T \left(\sum_j \int_0^{t_0} \xi_j(u)\,du H_{\epsilon_j}\right) \diamond \left(\sum_j \xi_j(t) H_{\epsilon_j}\right) dt \\
&= \int_0^T \sum_{j,k} \left(\int_0^{t_0} \xi_j(u)\,du \xi_j(t)\right) H_{\epsilon_j k}\,dt \\
&= \sum_{j,k} \int_0^{t_0} \xi_j(u)\,du \cdot \int_0^T \xi_j(t) H_{\epsilon_{jk}}\,dt \\
&= \left(\sum_j \int_0^{t_0} \xi_j(u)\,du H_{\epsilon_j}\right) \diamond \left(\sum_j \int_0^T \xi_j(u)\,du H_{\epsilon_j}\right) \\
&= B(t_0) \diamond B(T) \\
&= B(t_0) \cdot B(T) - (\mathbf{1}_{[0,t_0)}, \mathbf{1}_{[0,T)}) \\
&= B(t_0) \cdot B(T) - t_0
\end{aligned}$$

Note that we could have used Cor. 2.5.12, p. 54, to obtain this result immediately.

Answer to b: We calculate,

$$\begin{aligned}
\int_0^T \int_0^T g(s)\,dB(s)\,\delta B(t) &= \int_0^T (\int_0^T g(s)\,dB(s)) \diamond W(t)\,dt \\
&= \int_0^T g(s)\,dB(s) \diamond \int_0^T W(t)\,dt \\
&= B(T) \diamond \int_0^T g(s)\,dB(s)
\end{aligned}$$

By theory on p. 15,

$$\int_0^T g(s)\,dB(s) = \int_{I\!R} \mathbf{1}_{[0,T]}(s)\,dB(s) = \langle \omega, \mathbf{1}_{[0,T]} g \rangle$$

Thus, by exercise 1 (2.9) above,

$$\begin{aligned}
\int_0^T \int_0^T g(s)\,dB(s)\,\delta B(t) &= B(T) \diamond \int_0^T g(s)\,dB(s) \\
&= B(T) \int_0^T g(s)\,dB(s) - (\mathbf{1}_{[0,T]}, \mathbf{1}_{[0,T]} g) \\
&= B(T) \int_0^T g(s)\,dB(s) - \int_0^T g(s)\,ds
\end{aligned}$$

Answer to c: Since $B^{\diamond 2}(t_0) + t_0 = B^2(t_0)$,

$$\begin{aligned}\int_0^T B^2(t_0)\,\delta B(t) &= \int_0^T B^2(t_0) \diamond W(t)\,dt \\ &= B^2(t_0) \diamond B(T) \\ &= B^{\diamond 2}(t_0) \diamond B(T) + t_0 B(t_0)\end{aligned}$$

From exercise 1 (2.9) above,

$$B^{\diamond 2}(t_0) \diamond B(T) = B^2(t_0)B(T) - t_0 B(T) - 2t_0 B(t_0)$$

Altogether,

$$\begin{aligned}\int_0^T B^2(t_0)\,\delta B(t) &= B^2(t_0)B(T) - t_0 B(T) - 2t_0 B(t_0) + t_0 B(T) \\ &= B^2(t_0)B(T) - 2t_0 B(t_0)\end{aligned}$$

Day 3

Exercise 1:

Answer to 2.8 e: By (2.6.48), p. 65,

$$\begin{aligned}\exp^{\diamond}(w(\eta_1)) &= \sum_{n=0}^{\infty} \frac{1}{n!} w^{\diamond n}(\eta_1) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} h_n(w(\eta_1)) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} h_n(\langle \omega, \eta_1 \rangle)\end{aligned}$$

where we have used (2.4.17), p.44. From def. 2.2.1, p.19,

$$H_\alpha(\omega) = \Pi_{i=1}^{\infty} h_{\alpha_i}(\langle \omega, \eta_1 \rangle)$$

Thus

$$H_{n\epsilon_1}(\omega) = h_n(\langle \omega, \eta_1 \rangle)$$

which implies,

$$\exp^{\diamond}(w(\eta_1)) = \sum_{n=0}^{\infty} \frac{1}{n!} H_{n\epsilon_1}(\omega)$$

Answer to 2.8 f: From (2.6.55), p. 66,

$$\exp^{\diamond}(w(\eta_1) = \exp(w(\eta_1)) \exp(-1/2)$$

By exercise above we get,

$$\exp(w(\eta_1)) = \sum_{n=0}^{\infty} \frac{\sqrt{e}}{n!} H_{n\epsilon_1}$$

Answer to 2.11: From (2.6.54), p. 66,

$$\begin{aligned}\mathrm{E}\left[K_\phi(x)\right] &= \mathrm{E}\left[\exp^\diamond(W_\phi(x))\right]\\ &= \exp\left(\mathrm{E}\left[W_\phi(x)\right]\right)\\ &= \exp(0) = 1\end{aligned}$$

By (2.6.56), p. 67,

$$K_\phi(x,\omega) = \exp\left(W_\phi(x,\omega) - \frac{1}{2}||\phi||^2\right)$$

Thus,

$$\begin{aligned}K_\phi^2(x) &= \exp\left(2W_\phi - ||\phi||^2\right)\\ &= \exp\left(W_{2\phi} - \frac{1}{2}||2\phi||^2\right)\cdot\exp\left(\frac{1}{2}||2\phi||^2 - ||\phi||^2\right)\\ &= \exp^\diamond(W_{2\phi})\cdot\exp(||\phi||^2)\end{aligned}$$

We can calculate,

$$\begin{aligned}\mathrm{Var}\left[K_\phi(x)\right] &= \mathrm{E}\left[K_\phi^2(x)\right] - \mathrm{E}\left[K_\phi(x)\right]^2\\ &= \exp(||\phi||^2)\cdot\exp(\mathrm{E}[W_{2\phi}]) - 1\\ &= \exp||\phi||^2 - 1\end{aligned}$$

Answer to 2.13: Define

$$\cos^\diamond X = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}X^{\diamond 2n}$$

Let $X = w(\phi)$ and apply the Hermite transform: From def. 2.6.14, p. 65,

$$\left(\cos^\diamond w(\phi)\right)^{\tilde{}}(z) = \cos(\tilde{w}(\phi)(z)) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\tilde{w}(\phi)(z)\right)^{2n}$$

We know from calculus that

$$\cos(\tilde{w}(\phi)(z)) = \frac{1}{2}\left(\exp(i\tilde{w}(\phi)(z)) + \exp(-i\tilde{w}(\phi)(z))\right)$$

By inverting the Hermite transform,

$$\begin{aligned}\cos^\diamond(w(\phi)) &= \frac{1}{2}\left(\exp^\diamond(iw(\phi)) + \exp(-iw(\phi))\right)\\ &= \frac{1}{2}\left(\exp^\diamond(w(i\phi)) + \exp^\diamond(w(-i\phi))\right)\end{aligned}$$

From (2.6.55), p. 66,

$$\exp^\diamond(w(i\phi)) = \exp\left(w(i\phi) - \frac{1}{2}||i\phi||^2\right)$$

By definition of the real $L^2(\mathbb{R})$-norm:

$$||i\phi||^2 = \int_{\mathbb{R}} (i\phi(x))^2\, dx = i^2 \int_{\mathbb{R}} \phi(x)^2\, dx = -||\phi||^2$$

Hence,

$$\exp(\frac{1}{2}||i\phi||^2) = \exp(\frac{1}{2}||\phi||^2)$$

Similarly, $|| - i\phi||^2 = -||\phi||^2$ and

$$\exp(-\frac{1}{2}|| - i\phi||^2) = \exp(\frac{1}{2}||\phi||^2)$$

We get,

$$\begin{aligned}\cos^{\diamond}(w(\phi)) &= \exp(\frac{1}{2}||\phi||^2)\frac{1}{2}\left(\exp(iw(\phi)) + \exp(-iw(\phi))\right) \\ &= \exp(\frac{1}{2}||\phi||^2)\cos w(\phi)\end{aligned}$$

Proof of b is similar.

Answer to 2.15 a: Since,

$$\frac{1}{2}(x-t)^2 = \frac{1}{2}x^2 - xt + \frac{1}{2}t^2$$

and

$$\frac{1}{2}x^2 - \frac{1}{2}(x-t)^2 = xt - \frac{1}{2}t^2$$

we get

$$\exp(tx - \frac{1}{2}t^2) = \exp(\frac{1}{2}x^2)\exp(-\frac{1}{2}(x-t)^2)$$

Taylor expansion of the last term around $t = 0$ gives,

$$\begin{aligned}\exp(-\frac{1}{2}(x-t)^2) &= \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^n}{dt^n}\exp(-\frac{1}{2}(x-t)^2)\,|_{t=0}\ \cdot t^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^n}{du^n}\exp(-\frac{1}{2}u^2)\,|_{u=x}\cdot(-1)^n\cdot t^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\left(\frac{d^n}{du^n}\exp(-\frac{1}{2}u^2)\right)\cdot t^n\end{aligned}$$

Invoking def. (C.1.), p. 207, we get

$$\exp(-\frac{1}{2}(x-t)^2) = \exp(-\frac{1}{2}x^2)\sum_{n=0}^{\infty}\frac{t^n}{n!}h_n(x)$$

Thus,

$$\exp(tx - t^2/2) = \sum_{n=0}^{\infty}\frac{t^n}{n!}h_n(x)$$

Answer to 2.15 b: Put $t = ||\phi||$ and $x = w(\phi)/||\phi||$ in the formula from exercise *a* above:

$$\exp(w(\phi) - \frac{1}{2}||\phi||^2) = \sum_{n=0}^{\infty} \frac{||\phi||^2}{n!} h_n(\frac{w(\phi)}{||\phi||})$$

Answer to 2.15 c: The result follows immediately with $\phi = \mathbf{1}_{[0,t]}$ in exercise *b* above.

Answer to 2.15 d: By definition of the Wick exponential,

$$\exp^{\diamond} w(\phi) = \sum_{n=0}^{\infty} \frac{1}{n!} w^{\diamond n}(\phi)$$

Lemma 2.6.16, p. 66, yields,

$$\exp^{\diamond} w(\phi) = \exp(w(\phi) - \frac{1}{2}||\phi||^2)$$

Hence,

$$\sum_{n=0}^{\infty} \frac{1}{n!} w^{\diamond n}(\phi) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(||\phi||^n h_n(\frac{w(\phi)}{||\phi||}) \right)$$

By uniqueness of the chaos expansion,

$$w^{\diamond n}(\phi) = ||\phi||^n h_n(\frac{w(\phi)}{||\phi||})$$

Answer to 2.24 d: By cor. 2.5.12, p. 54,

$$\begin{aligned} \int_0^T \exp(B(T))\, \delta B(t) &= \exp(B(T)) \diamond \int_0^T W(t)\, dt \\ &= \exp^{\diamond}(B(T)) \exp(T/2) \diamond B(T) \\ &= \exp(T/2)\, (B(T) \diamond \exp^{\diamond}(B(T))) \end{aligned}$$

By def. og the Wick exponential and exercise 2.15 *d* above (with $\phi = \mathbf{1}_{[0,T]}$):

$$\begin{aligned} B(T) \diamond \exp^{\diamond}(B(T)) &= \sum_{n=0}^{\infty} \frac{1}{n!} B^{\diamond(n+1)}(T) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} ||\mathbf{1}_{[0,T]}||^{n+1} h_{n+1}(\frac{B(T)}{||\mathbf{1}_{[0,T]}||}) \end{aligned}$$

Hence, since $||\mathbf{1}_{[0,T]}|| = T^{1/2}$,

$$\int_0^T \exp(B(T))\, \delta B(t) = \exp(T/2) \sum_{n=0}^{\infty} \frac{T^{n+1/2}}{n!} h_{n+1}(\frac{B(T)}{\sqrt{T}})$$

Exercise 2:

Answer to 3.4 a: Rewrite the equation for X_t in integral form,

$$X_t = x + \int_0^t rX_s\, ds + \int_0^t \alpha X_s \diamond W_s\, ds$$

We apply the Hermite transform to this integral equation. Denote $\mathcal{H}X_t(z) = \tilde{X}_t(z)$, and note that $\mathcal{H}(X_s \diamond W_s)(z) = \mathcal{H}X_s(z) \cdot \mathcal{H}W_s(z)$:

$$\tilde{X}_t(z) = x + \int_0^t (r + \alpha\tilde{W}_s(z))\tilde{X}_s(z)\, ds$$

Observe that this is an ordinary deterministic (complex-valued) differential equation for each z.

$$\frac{d}{dt}\tilde{X}_t(z) = (r + \alpha\tilde{W}_t(z))\tilde{X}_t(z), \qquad \tilde{X}_0(z) = x$$

The solution is known to be

$$\tilde{X}_t(z) = x\exp(rt)\exp(\alpha \int_0^t \tilde{W}_s(z)\, ds)$$

Note that

$$\mathcal{H}\left(\exp^{\diamond}(\alpha B(t))\right)(z) = \exp(\alpha \int_0^t \tilde{W}_s(z)\, ds)$$

Hence,

$$\begin{aligned} X_t &= x\exp(rt)\exp^{\diamond}(\alpha B(t)) \\ &= x\exp(rt)\exp(\alpha B(t) - \alpha^2 t/2) \\ &= x\exp((r - \alpha^2/2)t + \alpha B(t)) \end{aligned}$$

Answer to 3.4 b: The integral form of the equation is

$$X_t = x + \int_0^t rX_s\, ds + \int_0^t \alpha \diamond W_s\, ds$$

Apply the Hermite transform to get the differential equation

$$\frac{d}{dt}\tilde{X}_t(z) - r\tilde{X}_t(z) = \alpha\tilde{W}_t(z)$$

If we multiply both sides with the integrating factor $\exp(-rt)$ and integrate, we get

$$\exp(-rt)\tilde{X}_t(z) - \tilde{X}_0(z) = \int_0^t \alpha\exp(-rs)\tilde{W}_s(z)\, ds$$

Or, by rewriting and inverting the Hermite transform,

$$\begin{aligned} \tilde{X}_t &= x\exp(rt) + \exp(rt)\int_0^t \exp(-rs)\tilde{W}_s(z)\, ds \\ &= x\exp(rt) + \int_0^t \exp(r(t-s))\tilde{W}_s(z)\, ds \\ &= x\exp(rt) + \mathcal{H}\left(\int_0^t \exp(r(t-s))W_s\, ds\right)(z) \\ &= x\exp(rt) + \mathcal{H}\left(\int_0^t \exp(r(t-s)\, dB(s)\right)(z) \end{aligned}$$

We can conclude,

$$X_t = x\exp(rt) + \int_0^t \exp(r(t-s))\,dB(s)$$

Answer to 3.4 c: The integral form of the problem is,

$$X_t = x + \int_0^t r\,ds + \int_0^t \alpha X_s \diamond W(s)\,ds$$

Hermite transformation,

$$\tilde{X}_t(z) = x + rt + \int_0^t \alpha \tilde{X}_s(z)\tilde{W}_s(z)\,ds$$

which gives the differential equation,

$$\frac{d}{dt}\tilde{X}_t(z) - \alpha\tilde{W}_t(z)\tilde{X}_t(z) = r$$

Multiplying both sides with $\exp(-\alpha\int_0^t \tilde{W}_s(z)\,ds)$ and then integrating, yields,

$$\tilde{X}_t(z)\exp(-\alpha\int_0^t \tilde{W}_s(z)\,ds) - \tilde{X}_0(z) = \int_0^t r\exp(-\alpha\int_0^s \tilde{W}_u(z)\,du)\,ds$$

The Hermite transformed solution is:

$$\tilde{X}_t(z) = x\exp(\alpha\int_0^t \tilde{W}_s(z)\,ds) + \int_0^t r\exp(\int_s^t \tilde{W}_u(z)\,du)\,ds$$

Observe that

$$\mathcal{H}\left(\exp^{\diamond}(\alpha(B(t)-B(s))\right)(z) = \exp(\alpha\int_0^t \tilde{W}_u(z)\,du)$$

Inversion of the Hermite transform provides us with the solution:

$$\begin{aligned} X_t &= x\exp^{\diamond}(\alpha B(t)) + \int_0^t r\exp^{\diamond}(\alpha(B(t)-B(s)))\,ds \\ &= x\exp(\alpha B(t) - \alpha^2 t/2) + \int_0^t r\exp(\alpha(B(t)-B(s)) + \alpha^2(t-s)/2)\,ds \end{aligned}$$

Answer to 3.5 a: Rewriting the problem on integral form:

$$X_t = G + \int_0^t rX_s\,ds + \int_0^t \alpha X_s \diamond W_s\,ds$$

Hermite transform,

$$\tilde{X}_t(z) = \tilde{G}(z) + \int_0^t r\tilde{X}_s(z)\,ds + \int_0^t \alpha\tilde{X}_s(z)\tilde{W}_s(z)\,ds$$

or,

$$\frac{d}{dt}\tilde{X}_t(z) = (r + \alpha\tilde{W}_t(z))\tilde{X}_t(z)$$

Solving as in the exercises above,

$$\tilde{X}_t(z) = \tilde{G}(z) \exp(rt + \alpha \int_0^t \tilde{W}_s(z)\, ds)$$

Inverting the Hermite transform, using the fact $\mathcal{H}(X \diamond Y) = \mathcal{H}X\mathcal{H}Y$,

$$\begin{aligned} X_t &= G \diamond \exp^{\diamond}(rt + \alpha B(t)) \\ &= G \diamond \exp((r - \alpha^2/2)t + \alpha B(t)) \end{aligned}$$

Answer to 3.5 d: Rewriting,

$$X_t = G + \int_0^t B(T)\, dt + \int_0^t X_s \diamond W_s\, ds$$

Hermite transformation,

$$\tilde{X}_t(z) = \tilde{G}(z) + \tilde{B}_T(z)t + \int_0^t \tilde{X}_s(z)\tilde{W}_s(z)\, ds$$

or,

$$\frac{d}{dt}\tilde{X}_t(z) - \tilde{W}_t(z)\tilde{X}_t(z) = \tilde{B}_T(z)$$

Multiplying with the integrating factor $\exp(-\int_0^t \tilde{W}_s(z)\, ds)$ and integrating,

$$\tilde{X}_t(z)\exp(-\int_0^t \tilde{W}_s(z)\, ds) - \tilde{X}_0(z) = \int_0^t \exp(-\int_0^s \tilde{W}_u(z)\, du)\tilde{B}_T(z)\, ds$$

Inversion of the Hermite transform yields the solution,

$$X_t = G \diamond \exp^{\diamond}(B(t)) + B(T) \diamond \int_0^t \exp^{\diamond}(B(t) - B(s))\, ds$$

Day 4

Exercise 1:

Answer to 4.4 a: Consider the problem

$$\begin{cases} \frac{\partial U}{\partial t} = LU + U \diamond W_\phi(x,\omega), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d \\ U(0,x) = f(x) & x \in \mathbb{R}^d \end{cases}$$

Denote the Hermite transform of $U(t,x)$ by $\tilde{U}(t,x;z)$. By formally applying the Hermite transform to the problem above, we get a deterministic partial differential equation of parabolic type (with complex values):

$$\begin{cases} \frac{\partial \tilde{U}}{\partial t} = L\tilde{U} + \tilde{U} \diamond \tilde{W}_\phi(z), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d \\ \tilde{U}(0,x;z) = f(x) & x \in \mathbb{R}^d \end{cases} \tag{3.1}$$

By considering the real and imaginary part of $\tilde{U}$, we have a Feynman-Kac representation of the solution (see e.g. [KS]),

$$\tilde{U}(t,x;z) = \hat{E}^x\left[f(X_t)\exp(\int_0^t \tilde{W}_\phi(X_s;z)\, ds)\right]$$

where $\hat{E}^x$ is the expectation with respect to the probability $\hat{P}$ on the space $\hat{\Omega}$ (where the process X_t lives), conditioned on $X_0 = x$. For each z, $\tilde{U}(t,x;z)$ solves (3.1). We must prove the following,

1 $U(t,x) = \mathcal{H}^{-1}(\tilde{U}(t,x;z))$ for a $U(t,x) \in (\mathcal{S})_{-1}$.

2 that we can invert the Hermite transform through (3.1), i.e.

$$\mathcal{H}\frac{\partial U}{\partial t}(z) = \frac{\partial \tilde{U}(z)}{\partial t}$$
$$\mathcal{H}LU(z) = L\tilde{U}(z)$$

We consider 1, which we divide into three subparts:
(i): We prove that $s \to \tilde{W}_\phi(X_s(\hat{\omega});z)$ is continuous on $[0,t]$ for each $z \in K_q(R)$ and bounded in (s,z) on $[0,t] \times K_q(R)$ (almost surely $\hat{\omega}$).

Since (ϕ_x, η_j) is the convolution between $\phi \in L^2(\mathbb{R}^d)$ and $\eta_j \in \mathcal{S}(\mathbb{R}^d)$, the function $x \to (\phi_x, \eta_j)$ is bounded and uniformly continuous. Consider $x \to \tilde{W}_\phi(x;z)$ for $z \in K_q(R)$ (for some q, R):

$$\tilde{W}_\phi(x;z) = \lim_{N\to\infty} \sum_{j=1}^{N} (\phi_x, \eta_j) z_j$$

and thus $\tilde{W}_\phi(x;z)$ is the limit of a uniformly convergent sequence of uniformly continuous functions. Hence, $\tilde{W}_\phi(x;z)$ is continuous in x for $z \in K_q(R)$. From stochastic analysis we know that $s \to X_s(\hat{\omega})$ is continuous for almost all paths. Thus, $s \to \tilde{W}_\phi(X_s(\hat{\omega});z)$ is continuous for $z \in K_q(R)$
(ii): We show that $\tilde{W}_\phi(X_s(\hat{\omega});z)$ is integrable (in s) in $(\mathcal{S})_{-1}$ and

$$\mathcal{H}\left(\int_0^t W_\phi(X_s(\hat{\omega}), \omega)\, ds\right)(z) = \int_0^t \tilde{W}_\phi(X_s(\hat{\omega}); z)\, ds$$

By (i) above, all assumptions in lemma 2.8.5, p. 78, are satisfied for

$$\mathcal{H}(W_\phi(X_s(\hat{\omega}), \cdot))(z) = \tilde{W}_\phi(X_s(\hat{\omega}); z)$$

Hence, (ii) follows.
(iii): We show that

$$\mathcal{H}\left(\hat{E}^x\left[f(X_t)\exp^\diamond(\int_0^t W_\phi(X_s,\omega)\, ds)\right]\right)(z) = \hat{E}^x\left[f(X_t)\exp(\int_0^t \tilde{W}_\phi(X_s;z)\, ds)\right]$$

Note that $\hat{E}^x$ is taken with respect to $\hat{\omega}$, the variable in X_t. What we need to prove is that the Hermite transform commutes with $\hat{E}^x$. We haven't specified any topology for the probability space on which X_t lives, and thus we cannot say anything about the continuity of the mapping $\hat{\omega} \to X_t(\hat{\omega})$. Hence, lemma 2.8.5, p. 78 cannot be used. But there exists a weaker result on integration implying that the functional in question is Pettis integrable (rather than integrable in the sense of Bochner which is the essence of lemma 2.8.5.). We don't state the general result, but simply give the conditions for our special case: If

$$\hat{\omega} \to f(X_t(\hat{\omega}))\exp(\int_0^t \tilde{W}_\phi(X_s(\hat{\omega}); z)\, ds)$$

is measurable and bounded on $\hat{\Omega}$ for every $z \in K_q(R)$ for some positive q, K, then

$$f(X_t(\hat{\omega}))\exp^{\diamond}(\int_0^t W_\phi(X_s(\hat{\omega}),\omega)\,ds$$

is Pettis integrable on $\hat{\Omega}$ and the Hermite transform commutes with $\hat{E}^x$. But we can find q, K positive such that

$$\hat{\omega} \to f(X_t(\hat{\omega}))\exp(\int_0^t \tilde{W}_\phi(X_s(\hat{\omega});z)\,ds)$$

is bounded for every $z \in K_q(R)$ since $f \in L^\infty(\mathbb{R}^d)$ and $\phi \in L^2(\mathbb{R}^d)$. Thus we have proved the result.

Summing up, we can conclude

$$U(t,x) = \hat{E}^x\left[f(X_t)\exp^{\diamond}(\int_0^t W_\phi(X_s,\omega)\,ds)\right]$$

which proves part 1.

Part 2 can be proven using lemma 2.8.4, p. 77 together with regularity of $\tilde{U}(t,x;z)$ in (t,x), due to the fact that L is uniformly elliptic (which essentially makes $\tilde{U}(t,x;z)$ $C^{1,2}$ for every $z \in K_q(R)$.) We skip this proof here, and leave it to the interested reader.

Answer to 4.4 b: Since $W_\phi(x) \to W_x$ in $(\mathcal{S})_{-1}$ when $\phi \to \delta_0$. Therefore, when $\phi \to \delta_0$,

$$\exp^{\diamond}(\int_0^t W_\phi(X_s,\omega)\,ds) \to \exp^{\diamond}(\int_0^t W_{X_s}\,ds)$$

in $(\mathcal{S})_{-1}$ a.s–$\hat{\Omega}$ (this is true since we have combined an analytic function exp() with a convergent sequence). We can conclude

$$\hat{E}^x\left[f(X_t)\exp^{\diamond}(\int_0^t W_\phi(X_s,\omega)\,ds)\right] \to \hat{E}^x\left[f(X_t)\exp^{\diamond}(\int_0^t W_{X_s}\,ds)\right]$$

in $(\mathcal{S})_{-1}$ when $\phi \to \delta_0$.

Exercise 2:

From cor. 4.3.2, p. 150 with $K(t,x) = W_t$ and $\sigma = 1$,

$$\begin{aligned} U(t,x) &= \hat{E}^x\left[f(b_t)\diamond\exp^{\diamond}(\int_0^t W_{t-s}\,ds)\right] \\ &= \hat{E}^x\left[f(b_t)\diamond\exp(\int_0^t W_s\,ds)\right] \\ &= \hat{E}^x\left[f(B_t)\right]\diamond\exp(B(t)-t/2) \end{aligned}$$

Exercise 3:

From the assumptions on f, $u(t,x) \in (\mathcal{S})_{-1}$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$. Moreover, $u(t,x)$ is an $(\mathcal{S})_{-1}$-solution of

$$\frac{\partial u}{\partial t} = \frac{1}{2}\triangle u, \quad u(0,x) = f(x)$$

Obviously $X_t \in L^2(\mu)$ since $X_t = \exp(B(t) - t/2)$, and thus $U(t,x) = u(t,x) \diamond X_t \in (\mathcal{S})_{-1}$. By Wick Calculus we have the following:

$$\begin{aligned}
\frac{\partial}{\partial t}(u(t,x) \diamond X_t) &= \frac{\partial u}{\partial t}(t,x) \diamond X_t + u(t,x) \diamond \frac{\partial}{\partial t} X_t \\
&= \left(\frac{1}{2}\triangle u(t,x)\right) \diamond X_t + u(t,x) \diamond X_t \diamond W_t \\
&= \frac{1}{2}\triangle(u(t,x) \diamond X_t) + (u(t,x) \diamond X_t) \diamond W_t \\
&= \frac{1}{2}\triangle U(t,x) + U(t,x) \diamond W_t
\end{aligned}$$

Moreover, $U(0,x) = u(0,x) \diamond X_0 = f(x) \diamond 1 = f(x)$.

Day 5

Exercise 1:

Answer to 4.5: Given

$$U(t,x) = \frac{1}{2}\int_0^t \int_{x+s-t}^{x+t-s} W(s,y)\, dy\, ds$$

The proof that U is the unique $(\mathcal{S})^*$-solution of the problem will go as follows:

1 $U(t,x) \in (\mathcal{S})^*$

2 $\mathcal{H}U(t,x)(z) := \tilde{U}(t,x;z)$ is the unique solution of

$$\begin{cases} \frac{\partial^2 \tilde{U}(t,x;z)}{\partial t^2} - \frac{\partial^2 \tilde{U}(t,x;z)}{\partial x^2} = \tilde{W}(t,x;z), & (t,x) \in \mathbb{R}_+ \times \mathbb{R} \\ \tilde{U}(0,x;z) = \frac{\partial \tilde{U}}{\partial t}(0,x;z) = 0 & \end{cases}$$

for every $z \in \mathbb{C}_c^{\mathbb{N}}$.

3 The Hermite transform commutes with differentiation, i.e

$$\mathcal{H}\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 \tilde{U}}{\partial t^2} \text{ and } \mathcal{H}\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 \tilde{U}}{\partial x^2}$$

We start with 1: The chaos expansion of $W(s,y)$ is

$$W(s,y) = \sum_{j=1}^{\infty} \eta_j(s,y) H_{\epsilon_j}$$

which implies that

$$U(t,x) = \sum_{j=1}^{\infty} \left(\frac{1}{2}\int_0^t \int_{x+s-t}^{x+t-s} \eta_j(s,y)\, dy\, ds\right) H_{\epsilon_j}$$

(since $W(s,y)$ is strongly integrable in $(\mathcal{S})^*$). Since

$$\eta_j(s,y) = (\xi_{\delta_1^{(j)}} \widehat{\otimes} \xi_{\delta_2^{(j)}})(s,y)$$

(see p. 19, (2.2.8) for notation), we can bound the coefficients by applying prop. 2.3.7, p.36: $U(t,x) \in (\mathcal{S})^*$ if

$$\sup_j \left(\frac{1}{2}\int_0^t \int_{x+s-t}^{x+t-s} \eta(s,y)\, dy\, ds\right)^2 (2j)^{-q} < \infty$$

for some $q > 0$. But,

$$\begin{aligned}\sup_{(s,y)\in \mathbb{R}^2} |\eta_j(s,y)| &\le \frac{1}{2}\sup_s |\xi_{\delta_1^{(j)}}(s)| \sup_y |\xi_{\delta_2^{(j)}}(y)| \\ &\quad + \frac{1}{2}\sup_s |\xi_{\delta_2^{(j)}}(s)| \sup_y |\xi_{\delta_1^{(j)}}(y)| \\ &\le K(\delta_1^{(j)})^{-1/12}(\delta_2^{(j)})^{-1/12} \\ &\le K\end{aligned}$$

Hence we get the bound

$$\left(\int_0^t \int_{x+s-t}^{x+t-s} \eta(s,y)\, dy\, ds\right)^2 \le K^2 (\int_0^t 2(t-s)\, ds)^2 = 4K^2t^2$$

Thus,

$$\sup_j K^2t^2(2j)^{-q} < \infty$$

for every $q > 0$. We have proved that $U(t,x) \in (\mathcal{S})^*$.
We consider part 2: The Hermite transform of $U(t,x)$ is seen to be

$$\tilde{U}(t,x;z) = \frac{1}{2}\int_0^t \int_{x+s-t}^{x+t-s} \tilde{W}(s,y;z)\, dy\, ds$$

where $\tilde{W}(s,y;z) = \sum_{j=1}^{\infty} \eta_j(s,y)z_j$ (we have assumed – which is also true – that the Hermite transform commutes with both integrals. Prove this!). Recall from calculus: $\partial/\partial t \int_0^t f(s,t)\, ds = f(t,t) + \int_0^t \partial/\partial t f(s,t)\, ds$:

$$\begin{aligned}\frac{\partial}{\partial t}\int_0^t \int_{x+s-t}^{x+t-s} \tilde{W}(s,y;z)\, dy\, ds &= \int_{x+t-t}^{x+t-t} \tilde{W}(t,y;z)\, dy + \\ &\quad + \int_0^t \frac{\partial}{\partial t}\int_{x+s-t}^{x+t-s} \tilde{W}(s,y;z)\, dy\, ds \\ &= \int_0^t \left(\tilde{W}(s,x+t-s;z) - \tilde{W}(s,x+s-t;z)\right) ds\end{aligned}$$

A similar calculation gives,

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\int_0^t \int_{x+s-t}^{x+t-s} \tilde{W}(s,y;z)\, dy\, ds &= 2\tilde{W}(t,x;z) + \\ &\quad + \int_0^t \left(\frac{\partial \tilde{W}}{\partial x}(s,x+t-s;z) - \frac{\partial \tilde{W}}{\partial x}(s,x+t-s;z)\right) ds\end{aligned}$$

On the other hand,

$$\frac{\partial^2}{\partial x^2}\int_0^t \int_{x+s-t}^{x+t-s} \tilde{W}(s,y;z)\,dy\,ds = \int_0^t \frac{\partial}{\partial x}\left(\tilde{W}(s,x+t-s;z) - \tilde{W}(s,x+s-t)\right) ds$$
$$= \int_0^t \left(\frac{\partial \tilde{W}}{\partial x}(s,x+t-s;z) - \frac{\partial \tilde{W}}{\partial x}(s,x+s-t;z)\right) ds$$

Hence,

$$\frac{\partial^2 \tilde{U}}{\partial t^2}(t,x;z) - \frac{\partial^2 \tilde{U}}{\partial x^2}(t,x;z) = \tilde{W}(t,x;z)$$

for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. Boundary conditions,

$$\tilde{U}(0,x;z) = 0, \quad \frac{\partial \tilde{U}}{\partial x}(0,x;z) = 0$$

We have proved part 2.

Consider part 3: We shall not investigate this point in detail, but refer to lemma 2.8.4, p. 77. This lemma however, is for processes in $(\mathcal{S})_{-1}$.The analogue for $(\mathcal{S})^*$ can be obtained by substituting the boundedness condition in z with an exponential growth condition. If we, however, want to show that $U(t,x)$ is an $(\mathcal{S})_{-1}$-solution, we can apply lemma 2.8.4 as it stands. Sketch of such a proof:

(i) Show that $\partial\tilde{U}/\partial t$ and $\partial\tilde{U}/\partial x$ are continuous in t and x and bounded in $(t,x,z) \in \mathbb{R}_+ \times \mathbb{R} \times K_q(R)$ for some q, R.

(ii) Show the same for $\partial^2\tilde{U}/\partial t^2$ and $\partial^2\tilde{U}/\partial x^2$.

From lemma 2.8.4 we can then conclude that differentiation and the Hermite transform commutes.

Answer to 4.4 b: If $W(s,y) = W(y)$, we have

$$U(t,x) = \frac{1}{2}\int_0^t \int_{x+s-t}^{x+t-s} W(y)\,dy\,ds = \frac{1}{2}\int_0^t (B(x+t-s) - B(x+s-t))\,ds$$

by the relation $B(t) = \int_0^t W(x)\,dx$.

Answer to 4.8: Notice that $c_0(x) = E[p(x)]$. Thus, using $E[X \diamond Y] = E[X]E[Y]$ we get

$$\begin{aligned} E\left[(K(x) \diamond p'(x))'\right] &= \left(E\left[K(x)\right]E\left[p'(x)\right]\right)' \\ &= \left(\exp E[W_\phi(x)] \cdot c_0'(x)\right)' \\ &= c_0''(x) \\ &= -1 \end{aligned}$$

Since $E[p(a)] = c_0(a) = c_0(b) = E[p(b)] = 0$, we have that $c_0(x)$ is the solution of the following second order differential equation:

$$\begin{cases} c_0''(x) = -1, & x \in [a,b] \\ c_0(a) = c_0(b) = 0 \end{cases}$$

It is straightforward to check that the solution is

$$c_0(x) = -\frac{1}{2}\left(x^2 - (a+b)x + ab\right)$$

$c_0(x)$ is the best *ω-constant approximation* to $p(x,\omega)$.

We proceed with $c_{\epsilon_j}(x)$: Introduce the short-hand notation c_j for c_{ϵ_j}. From (4.6.28), p. 172:

$$\frac{1}{2}c_j''(x) + \left(\int \phi(y-x)\xi_j(y)\,dy\right)' c_0'(x) = 0$$

for $x \in [a,b]$. We have used that $f = -1$ only gives chaos of order 0, namely -1. In addition we have the boundary conditions $c_j(a) = c_j(b) = 0$. Inserting the expression for c_0:

$$\frac{1}{2}c_j''(x) - (\phi_x', \xi_j) \cdot (-\frac{1}{2})(2x - (a+b)) = 0$$

or

$$c_j''(x) + (\phi_x', \xi_j)(2x - (a+b)) = 0$$

Define the function $g_j(x) = (\phi_x', \xi_j)(2x - (a+b))$. We can write the solution $c_j(x)$ on the following form,

$$c_j(x) = k_1 + k_2 x - \int_a^x \int_a^y g_j(z)\,dz\,dy$$

where the constants k_1 and k_2 are solutions of

$$k_1 + ak_2 = 0$$

and

$$k_1 + bk_2 = \int_a^b \int_a^y g_j(z)\,dz\,dy$$

The equations for k_1 and k_2 are obtained from the boundary conditions.

Note that $c_{\epsilon_j}(x)$ is the best *Gaussian approximation* to $p(x,\omega)$.

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