Long-range scattering of three-body quantum systems, I[1]

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Contents

1 Introduction and main result

This paper is the first in a series of two papers on asymptotic completeness for (generalized) three-body quantum systems with long-range interaction. The model to be studied is given in terms of a finite family of subspaces $\{X_a | a \in \mathcal{F}\}$ of a finite dimensional Euclidean space X. By definition $a_{\min}, a_{\max} \in \mathcal{F}$ are given by $X_{a_{\min}} = X$ and $X_{a_{\max}} = \{0\}$, respectively, and for a and b different from $a_{\min}$ the "three-body" condition $X_a \cap X_b = \{0\}$ is imposed. The position and momentum operators on the basic Hilbert space $\mathcal{H} = L^2(X)$ are denoted by x and p, respectively. The orthogonal complement of X_a in X is denoted by X^a. The corresponding components of x and p are denoted by x_a, p_a and x^a, p^a, respectively.

The basic Hamiltonian on $\mathcal{H}$ is

$$H = \tfrac{1}{2}p^2 + V; \; V(x) = \sum_{a \in \mathcal{F}} V^a(x^a), \tag{1.1}$$

where each "pair potential" V^a is assumed to be a real-valued smooth function on X^a obeying for some $\mu > 0$ (independent of a) and all multiindices β

$$\partial^{\beta}_{x^a} V^a(x^a) = O\Big(|x^a|^{-\mu-|\beta|}\Big). \tag{1.2}$$

Asymptotic completeness, henceforth denoted AC, for H is a characterization of the states in the continuous subspace in terms of simplified evolutions (see below for an account of the notion AC). It has been proved for three-body systems under the condition (1.2) for $\mu > \sqrt{3} - 1$ first by Enss ([E]) and then by a different method in the context of many-body scattering by Dereziński ([D]), and later for $\mu \in (\frac{1}{2}, \sqrt{3} - 1]$ under more conditions by Gérard and Wang ([G], [W]). The additional conditions imposed by Gérard are essentially spherical symmetry and a global virial condition. The latter implies among other properties a negative upper bound near infinity for each pair potential. The main result by Wang does not require spherical symmetry but essentially positivity near infinity for each pair potential. Finally Yafaev ([Y1]) constructed counterexamples for any $\mu \in \big(0, \frac{1}{2}\big)$ in systems of one-dimensional particles.

One may view the above results due to Gérard and Wang as supporting the conjecture that AC for H holds assuming only (1.2) for $\mu \in (\frac{1}{2}, \sqrt{3} - 1]$. We agree with a remark in [G] that indeed it would be very difficult to prove this conjecture. Of course it could be wrong but we see no indication of that. In view of Gérard's result the remark in [D] motivated by examples from classical mechanics with negative pair potentials near infinity, that $\mu > \sqrt{3} - 1$ "seems to be optimal", is disputable. On the other hand one may argue that the global virial condition of [G] is very restrictive even within classes of such pair potentials.

The aim of this paper is threefold: 1) We give a proof of AC for the regime $\mu \in (\frac{1}{2}, \sqrt{3}-1]$ under weaker conditions than in [G], explicitly without the global virial condition but keeping the negative upper bound near infinity. Consequently we provide further support for the above conjecture. 2) We give proofs relying entirely on so-called weak propagation estimates hence providing a considerable simplification compared to the extensive use of so-called strong propagation estimates in [G]. 3) We prove a structure theorem for the regime $\mu \in (0, \frac{1}{2}]$ constituting a basis for an AC result in this case. The latter is finally proved in the other paper in the series [S] by a rather different method compared to the approach for $\mu > \frac{1}{2}$ of this paper. In addition [S] contains a result on AC for positive potentials (at infinity) in the regime $\mu \in (0, \frac{1}{2}]$ (essentially proved by techniques of the present paper).

1.1 Asymptotic completeness

As explained by Dereziński in [D, Section 2] there are various equivalent notions of AC for many-body models and hence in particular for the above three-body model. We shall here recall two of these notions (specialized to our case). The first one relies on the concept of asymptotic velocity of [D], $D^+(H) = s - C_\infty - \lim_{t\to+\infty} e^{itH}t^{-1}xe^{-itH}$. We introduce for $a \in \mathcal{F}$ the notation

$$H^a = \frac{1}{2}(p^a)^2 + V^a,\ h_a = \frac{1}{2}p_a^2 + I_a;\ I_a = I_a(x_a) = V(x_a) - V^a(0).$$

The notation $E_\Omega(D)$ is used to denote the spectral projection corresponding to a Borel set Ω for some (possibly) vector-valued operator D of commuting self-adjoint operators. Let $E^{pp}(H^a)$ denote the projection onto the pure point subspace of H^a. Let $Z_a = X_a \backslash \big(\cup_{b\not\subset a} X_b\big)$, where by definition $b \subset a \Leftrightarrow X^b \subset X^a$.

Now we can consider the following first statement of existence of wave operators and AC: For all $a \neq a_{\max}$ there exists

$$W_{a1}^+ :\ \mathcal{H}_a = \ \mathrm{ran}E^{pp}(H^a) \otimes \mathrm{ran}E_{Z_a}\big(D^+(h_a)\big) \to \mathrm{ran}E_{Z_a}\big(D^+(H)\big),$$

$$W_{a1}^+ = s - \lim_{t\to+\infty} e^{itH}\Big(e^{-itH^a} \otimes e^{-ith_a}\Big).$$

Moreover

$$\text{AC1} \qquad \mathrm{ran}E^{pp}(H) \oplus \sum_{a\neq a_{\max}} \oplus\, \mathrm{ran}W_{a1}^+ = \mathcal{H}.$$

A more traditional statement of existence of wave operators and AC goes through the existence of a smooth real-valued function $S_a(t, \xi_a)$ on $\mathbf{R}^+ \times Z_a$ which for any compact $\Theta \subset Z_a$ satisfies

$$\sup_{\xi_a\in\Theta} |\partial_{\xi_a}^\beta(\nabla S_a(t,\xi_a) - t\xi_a)| \le C_\beta t^{1-\mu};\ t \ge 1,$$

$$\partial_t S_a(t,\xi_a) = \frac{1}{2}\xi_a^2 + I_a(\nabla S_a(t,\xi_a));\ \xi_a \in \Theta,\ t \ge t_\Theta.$$

The traditional statement reads: For all $a \neq a_{\max}$ there exists

$$W_{a2}^+ :\ \mathrm{ran}E^{pp}(H^a) \otimes L^2(X_a) \to \mathcal{H},$$

$$W_{a2}^+ = s - \lim_{t\to+\infty} e^{itH}\Big(e^{-itH^a} \otimes e^{-iS_a(t,p_a)}\Big).$$

Moreover

$$\text{AC2} \qquad \mathrm{ran}E^{pp}(H) \oplus \sum_{a\neq a_{\max}} \oplus\, \mathrm{ran}W_{a2}^+ = \mathcal{H}.$$

One virtue of AC1 is that this version of AC is invariant, while the virtue of AC2 is its content of simplified evolutions (although not canonical). The latter may serve as a basis for studying the scattering matrix. Henceforth the equivalent statements AC1 and AC2 are referred to as AC.

1.2 Main result

Two of our main results may be combined as follows:

Theorem 1.1 *Suppose in addition to* (1.2) *for some* $\mu \in (0, \sqrt{3}-1]$ *(required for all pair potentials), that for all* a *the potential* V^a *is spherically symmetric and for two positive numbers* c *and* R

(1.3) $\quad V^a(x^a) \leq -c|x^a|^{-\mu}, \; |x^a| \geq R.$

Let $\alpha = 2(2+\mu)^{-1}$. *Then for all* $a \neq a_{\max}, a_{\min}$ *there exists the projection*

(1.4) $\quad P_a^+ = s - \lim_{t\to+\infty} e^{itH} E_{[t^{\alpha-\epsilon}, t^{\alpha+\epsilon}]}(|x^a|) e^{-itH} E_{Z_a}\big(D^+(H)\big),$

which is independent of all small $\epsilon > 0$.

For any μ *as above there exist the limits* W_{a1}^+ *and*

$$\Omega_{a1}^+ : \; \big(I - P_a^+\big) E_{Z_a}\big(D^+(H)\big)\mathcal{H} \to \mathcal{H}_a,$$

$$\Omega_{a1}^+ = s - \lim_{t\to+\infty} \Big(e^{itH^a} \otimes e^{ith_a}\Big) e^{-itH}.$$

Furthermore $W_{a1}^+\mathcal{H}_a = \big(I - P_a^+\big) E_{Z_a}\big(D^+(H)\big)\mathcal{H}$, *and* $\Omega_{a1}^+ W_{a1}^+ = I$ *on* $\mathcal{H}_a$.

Under the further condition $\mu \in (\frac{1}{2}, \sqrt{3}-1]$

(1.5) $\quad P_a^+ = 0.$

In particular AC holds for this regime.

At a first glance the above result for $\mu \in (0, \frac{1}{2})$ may seem to contradict the counterexamples in [Y1] (for which exceptional states in $\Big\{x^a \in X^a \big| \, |x^a| \leq t^{\frac{1}{2}}\Big\}$ are constructed), but of course the geometric condition (1.3) is not compatible with these examples. Moreover we remark that (1.3) excludes the existence of zero-energy bound states for subsystems. As a consequence (using decay of subsystem bound states) one can give a simple direct proof of the existence of W_{a1}^+, cf. [D, Theorem 2.6 (b)].

Although we shall not elaborate in this paper the lower bound (1.3) can be slightly relaxed (by introducing a bigger μ in this condition), and one can also add a fast decaying possibly locally singular and non-spherical symmetric perturbation of each pair potential. Also we remark that for one-dimensional pair potentials we dont need spherical symmetry. The core of our methods is essentially one-dimensional, and for this reason we find it convenient to treat this case, or rather a simplified one-dimensional model, in detail while we only sketch the proof of Theorem 1.1 starting here with a discussion of a reduction procedure (see also the outline of the paper at the end of this section). As it follows we analyse for fixed a; the additional assumptions of Theorem 1.1 are only needed for the a in question.

1.3 A reduction

We shall briefly recall the reduction for AC of [D] (see also [E], for example). It constitutes the first step of the proof of the AC-statement of Theorem 1.1 (and for the general part as well). By an application of the Mourre estimate

$$\mathcal{H} = \mathrm{ran} E^{pp}(H) \oplus \sum_{a \neq a_{\max}} \oplus \, \mathrm{ran} E_{Z_a}\big(D^+(H)\big),$$

cf. [D, (4.17)]. To show AC1 we let any $a \neq a_{\max}, a_{\min}$ and $\phi \in \mathcal{H}$ be given. We need to show the existence of $\lim_{t\to+\infty} \big(e^{itH^a} \otimes e^{ith_a}\big) e^{-itH} E_{\Theta_a}\big(D^+(H)\big)\phi$ for any compact $\Theta_a \subset Z_a$. Now using [D, Proposition 4.7] there exists $\phi_a^+ = \lim_{t\to+\infty} U_a(t)^* e^{-itH} E_{\Theta_a}\big(D^+(H)\big)\phi$, where $U_a(t)$ is generated by $H^a + \frac{1}{2}p_a^2 + J\left(\frac{x}{t}\right)I_a(x)$ with J an arbitrary C_0^∞ cutoff function supported in $Y_a = X\backslash\left(\cup_{b\not\subset a} X_b\right)$ and equal to one on a neighborhood of Θ_a, and with $I_a(x) = V(x) - V^a(x^a)$. Thus it remains to show that there exists the limit

$$\lim_{t\to+\infty}\left(e^{itH^a}\otimes e^{ith_a}\right)U_a(t)\phi_a^+.$$

For that purpose we notice that there exists the asymptotic energy $H^{a+} = \lim_{t\to+\infty} U_a(t)^* H^a U_a(t)$ (understood in the strong resolvent sense). Next by another application of the Mourre estimate [D, Lemma 4.10] and the exponential decay of negative-energy bound states it suffices to consider $\phi_a^+ \in E_{\{0\}}\left(H^{a+}\right)\mathcal{H}$. Applying the chain rule again it suffices in fact to show the existence of the limit

$$\tilde{\phi}_a^+ = \lim_{t\to+\infty}\tilde{U}_a(t)^* U_a(t)\phi_a^+, \tag{1.6}$$

where $\tilde{U}_a(t)$ is generated by $H^a + \frac{1}{2}p_a^2 + J\left(\frac{x_a}{t}\right)I_a(x_a)$ and $\phi_a^+ \in E_{\{0\}}\left(H^{a+}\right)\mathcal{H}$.

We have completed an outline of the reduction procedure for AC for the three-body model. (A similar procedure would work for the existence of W_{a1}^+.) We notice that the potential $I_a(t,x) = J\left(\frac{x}{t}\right)I_a(x)$ obeys

$$\partial_x^\beta I_a(t,x) = O\left(t^{-\mu-|\beta|}\right) \text{ uniformly in } x. \tag{1.7}$$

Motivated by the above reduction we are going to suppress the variables x_a and p_a, for example only derivatives w.r.t. x^a (up to order one) will matter in (1.7). In fact in the bulk of this paper we shall consider only simplified models given by omitting (or "freezing") the variables x_a and p_a (cf. [D]). Of course this procedure is justified as long as the various constructions (for example propagation observables) to be considered are independent of the omitted variables.

1.4 Ideas and organization of proofs

Obviously this paper is to some extent dependent on [G]. However we would like stress that it is completely self-contained and relying entirely on weak propagation estimates. The latter agrees with most of the known proofs of AC for many-body systems for instance the one of [D] to which we refer the reader for an account of the history of AC (in particular for short-range systems which is not mentioned in this introduction). The weak propagation estimates of this paper are established in the standard fashion by constructing uniformly bounded families of observables whose Heisenberg derivative has a sign up to an integrable error, see for example [D, Lemma A.1, (b)].

Moreover an important intermediate step of our approach is absence of propagation in the region $t^{\alpha^2\mu+\epsilon} \leq |x^a| \leq t^{\alpha-\epsilon}$ (precisely this means that $||E_{\left[t^{\alpha^2\mu+\epsilon},t^{\alpha-\epsilon}\right]}(|x^a|)U_a(t)\phi_a^+|| \to 0$ for $\phi_a^+ \in E_{\{0\}}\left(H^{a+}\right)\mathcal{H}$), which follows from an entirely different method. (In [G] the global virial condition is used to exclude the entire inner region.) Although this method is somewhat complicated partly due to constructions needed to diminish impact of the uncertainty principle the basic idea is simple: For a state in the indicated region the radial part p_r^a of the momentum p^a cannot be negative (otherwise the wave packet would not propagate as fast as indicated). Moreover by an energy bound the corresponding kinetic energy $\frac{1}{2}(p_r^a)^2$ must be of size $-V^a(x^a)$ (provided that the angular momentum is small). Putting together $\frac{d}{dt}|x^a| = p_r^a \approx \sqrt{-2V^a(x^a)}$ for a state in this region, which classically is impossible since the equation is solved by $|x^a| \approx t^\alpha$. Our approach may be viewed as a quantum version of a similar but simpler approach in the corresponding classical model.

Similarly the absence of propagation in the region $|x^a| \geq t^{\alpha+\epsilon}$ can be understood classically. (We give a different proof than those in [D] and [G].) Thus we are left with the inner region $|x^a| \leq t^{\alpha^2\mu+\epsilon}$ and the "classical region" $t^{\alpha-\epsilon} \leq |x^a| \leq t^{\alpha+\epsilon}$ (the latter supporting the above solution) . For each of these separate regions quantum mechanics enters crucially in terms of certain wave operators (cf. Subsection 1.3). For the classical region the existence of the corresponding wave operator is only shown in this paper for $\mu > \frac{1}{2}$, in which case we conclude (under the conditions of Theorem 1.1) that in fact $E_{\{0\}}\big(H^{a+}\big) = 0$.

As indicated above this paper concerns a zero-energy problem for a time-dependent Hamiltonian. We refer to [N] and [Y2] for results on zero-energy problems for two-body time-independent long-range potentials.

This paper is organized as follows: We treat the simplified one-dimensional model in detail in Sections 2–6. It is introduced in Section 2, where also various results similar to Theorem 1.1 are stated and some preliminaries are given. In Sections 3–6 we treat various regions separately, cf. the discussion above: Those regions may be called the exterior, intermediate, classical and inner region, respectively. The combination of Theorem 6.3 and Corollary 6.4 is the direct analogue of Theorem 1.1. In Section 7 we treat a simplified multi-dimensional model with a spherically symmetric potential. Following the procedure of the previous sections we shall sketch a proof of Theorems 7.1 and 7.2 which in combination corresponds to Theorem 1.1.

We thank Andreas Hinz and Hubert Kalf for drawing our attention to the reference [O] for a result on absence of a zero-energy bound state for one-dimensional Hamiltonians (which made an appendix in a preliminary draft of this paper superfluous).

2 One-dimensional results and preliminaries

Let $\mu \in (0, \sqrt{3}-1]$ be fixed. On the Hilbert space $\mathcal{H} = L^2(\mathbf{R}_x)$ we consider a time-dependent Hamiltonian $H(t) = H + I(t),\ H = \frac{1}{2}p^2 + V(x),\ I(t) = I(t,x),\ t > 0$, where $p = -i\frac{d}{dx}$ denotes the momentum operator, $I(\cdot) \in C^0\big(\mathbf{R}_+, C^1(\mathbf{R}_x)\big)$ and $V \in C^\infty$ with

(2.1)
$$|\frac{d^k}{dx^k}I(t,x)| \le C_k t^{-\mu-k};\ k = 0,1,$$
$$\frac{d^k}{dx^k}V(x) = O\Big(|x|^{-\mu-k}\Big).$$

In addition $I(t,x)$ and $V(x)$ are real-valued.

We shall impose one or both of the following geometric conditions:

(2.2) $\quad V(x) \le -c|x|^{-\mu};\ x \ge R.$

(2.3) $\quad V(x) \le -c|x|^{-\mu};\ x \le -R.$

Here c and R are positive constants.

Throughout the paper the letters α and γ are used to denote

(2.4) $\quad \alpha = \frac{2}{2+\mu},\ \gamma = \frac{\mu}{2+\mu}.$

Under the condition (2.1) we can consider the propagator $U(t)$ obeying (at least formally)

(2.5) $\quad i\partial_t U(t) = H(t)U(t),\ U(1) = I.$

We refer to [DG, Appendix B.3] for an elaboration; it is known that $U(t)$ preserves the domain of p. We consider the corresponding asymptotic energy (the limit being understood in the strong resolvent sense)

(2.6) $\quad H^+ = \lim_{t\to+\infty} U(t)^* H U(t).$

Let $\phi^+ \in E_{\{0\}}(H^+)\mathcal{H}$ be arbitrarily given. We are aiming at completeness under (2.1) and the geometric conditions (2.2) and (2.3), which (by definition) amounts to showing that there exist

$$\text{(2.7)} \quad \tilde{\phi}^+ = \lim_{t\to+\infty} e^{i\int_1^t (H+I(s,0))ds} \phi^+(t) = \lim_{t\to+\infty} e^{i\int_1^t I(s,0)ds} \phi^+(t);\ \phi^+(t) = U(t)\phi^+.$$

Notice that indeed if the first limit of (2.7) exists then the second also exists, and they are the same and $\tilde{\phi}^+ \in \ker(H)$. Consequently, cf. [O, Theorem 2.2 p. 196], $\phi^+ = 0$.

In this paper the existence of (2.7) will be shown for $\mu \in (\frac{1}{2}, \sqrt{3}-1]$.

In general for $\mu \in (0, \sqrt{3}-1]$ we shall show the following preliminary result: We introduce

$$\text{(2.8)} \quad \begin{aligned} &P^+ = P_r^+ + P_l^+,\\ &P_r^+ = s-\lim_{t\to+\infty} U(t)^* E_{[t^{\alpha-\epsilon}, t^{\alpha+\epsilon}]}(x) U(t) E_{\{0\}}(H^+),\\ &P_l^+ = s-\lim_{t\to+\infty} U(t)^* E_{[-t^{\alpha+\epsilon}, -t^{\alpha-\epsilon}]}(x) U(t) E_{\{0\}}(H^+);\ \epsilon > 0, \end{aligned}$$

where the limits will be shown to exist and be independent of (small) $\epsilon > 0$. Then we shall show the existence of

$$\text{(2.9)} \quad \tilde{\phi}^+ = \lim_{t\to+\infty} e^{i\int_1^t I(s,0)ds} U(t)\big(I - P^+\big)\phi^+.$$

We note that without the conditions (2.2) and (2.3) the statement (2.9) has counterexamples for $\mu \in (0, \frac{1}{2})$, see [DG, Section 3.8.3].

Clearly the existence of (2.7) follows from the existence of (2.9) if $P^+ = 0$. The latter property will be proved for $\mu \in (\frac{1}{2}, \sqrt{3}-1]$ in this paper.

Imposing only the additional condition (2.2) (not (2.3) here) we shall prove that for any $\tilde{\alpha} > \alpha_{cr} := \alpha^2\mu$

$$\text{(2.10)} \quad \lim_{t\to+\infty} ||F\big(t^{-\tilde{\alpha}}x > 1\big)U(t)\big(I - P_r^+\big)\phi^+|| = 0.$$

Here we have used notation from the following list.

Definitions 2.1 *For any* $C > 0$ *the notation* $F(\cdot > C)$, $F(\cdot < C)$ *and* $F(\cdot \approx C)$ *is used to denote smooth functions with* $x^k \frac{d^k}{dx^k}F(x)$ *bounded for all* k *and supported in* $\big(\frac{1}{2}C, \infty\big)$, $(-\infty, 2C)$ *and* $\big(\frac{1}{2}C, 2C\big)$*, respectively.*

For $C < 0$ *the same notation is used with the meaning* $F(-\cdot < -C)$, $F(-\cdot > -C)$ *and* $F(-\cdot \approx -C)$*, respectively.*

The notation $F(C_1\cdot > 1, C_2\cdot < 1)$ *is used to symbolize the product function* $F(C_1\cdot > 1)F(C_2\cdot < 1)$ *(for* $0 < C_2 < C_1$*).*

Let $\mathcal{F}_+$ *denote the largest set of* $F = F_+ = F_+(\cdot > 1) = F(\cdot > 1)$*, such that (in addition)* $0 \le F \le 1$, $F' = \frac{d}{dx}F \ge 0, F' \in C_0^\infty\big(\big(\frac{1}{2}, 2\big)\big)$*,* $F\big(\frac{1}{2}\big) = 0$, $F(2) = 1$ *and* $\sqrt{1-F}$, $\sqrt{F}$, $\sqrt{F'} \in C^\infty$*, which is stable under the maps* $F \to F^m$ *and* $F \to 1-(1-F)^m$; $m \in \mathbf{N}$*. Let* $\mathcal{F}_-$ *denote the set of functions* $F_- = 1 - F_+$ *where* $F_+ \in \mathcal{F}_+$*.*

The proof of (2.10) is divided into two parts: We show in Sections 3 and 4 that for any $\epsilon > 0$

$$\text{(2.11)} \quad \lim_{t\to+\infty} ||F\big(\tfrac{x}{t^{\alpha+\epsilon}} > 1\big)\phi^+(t)|| = 0,$$

and

$$(2.12)\quad \lim_{t\to+\infty} ||F\big(\tfrac{x}{t^{\alpha_{cr}+\epsilon}} > 1, \tfrac{x}{t^{\alpha-\epsilon}} < 1)\big)\phi^+(t)|| \,=\, 0,$$

respectively.

Obviously (2.11) and (2.12) imply the existence of P_r^+ and (2.10).

For $\mu \in (\frac{1}{2}, \sqrt{3}-1]$ the statement $P_r^+ = 0$ follows from

$$(2.13)\quad \lim_{t\to+\infty} ||F\big(\tfrac{x}{t^{\alpha-\epsilon}} > 1, \tfrac{x}{t^{\alpha+\epsilon}} < 1\big)\phi^+(t)|| \,=\, 0;$$

the latter will be proved in Section 5.

The existence of (2.9) will be shown in Section 6 under the conditions (2.2) and (2.3).

In the remaining part of this section we don't use (2.2) and (2.3).

We have the following elementary preliminary result.

Lemma 2.2 *Let* $\tilde{\alpha} \geq 0$ *and* $\kappa, t \geq 1$. *Then for all* $w \in \mathbf{C}$

$$(2.14)\quad ||F\big(t^{-\tilde{\alpha}}x > 1\big)p(\kappa H - w)^{-1}|| \leq C\tfrac{\langle w\rangle^{\frac{1}{2}}}{|\mathit{Im} w|}\big(t^{-\mu\tilde{\alpha}} + \kappa^{-1}\big)^{\frac{1}{2}};\ \langle w\rangle \,=\, \big(1+|w|^2\big)^{\frac{1}{2}}.$$

Proof Let $\psi \in \mathcal{H}$ with $||\psi|| = 1$. Then

$$||p\tilde{\psi}||^2 = \kappa^{-1}\langle 2\kappa(H-V)\rangle_{\tilde{\psi}};\ \tilde{\psi} = (\kappa H - w)^{-1}\psi.$$

(Here and in the following $\langle A\rangle_{\psi}$ is used to denote the expectation value of A in a state ψ.) Next we rewrite

$$\langle 2\kappa(H-V)\rangle_{\tilde{\psi}} = 2\langle \kappa H - w\rangle_{\tilde{\psi}} + 2\langle w - \kappa V\rangle_{\tilde{\psi}}$$

and bound the first term on the right hand side as

$$2\langle \kappa H - w\rangle_{\tilde{\psi}} \leq 2|\mathrm{Im} w|^{-1},$$

and the second as

$$2\langle w - \kappa V\rangle_{\tilde{\psi}} \leq C\frac{\kappa + |w|}{|\mathrm{Im} w|^2}$$

uniformly w.r.t. ψ.

Putting together yields to

$$(2.15)\quad ||p\tilde{\psi}||^2 \,\leq\, C\tfrac{\langle w\rangle}{|\mathrm{Im} w|^2}.$$

Next we compute

$$(2.16)\quad \begin{aligned} &||F\big(t^{-\tilde{\alpha}}x > 1\big)p\tilde{\psi}||^2 \leq C_1 t^{-2\tilde{\alpha}}|\mathrm{Im} w|^{-2} + 4\langle H - V\rangle_{F(\cdot)\tilde{\psi}} \\ &\leq C_2 t^{-\mu\tilde{\alpha}}|\mathrm{Im} w|^{-2} + C_3|\mathrm{Im} w|^{-1}||HF(\cdot)\tilde{\psi}||. \end{aligned}$$

But since (by (2.15))

$$||HF(\cdot)\tilde{\psi}|| \le C_4\Big(t^{-2\tilde{\alpha}}|\mathrm{Im}w|^{-1} + t^{-\tilde{\alpha}}||p\tilde{\psi}|| + ||F(\cdot)H\tilde{\psi}||\Big)$$

$$\le C_5\left(t^{-2\tilde{\alpha}}|\mathrm{Im}w|^{-1} + t^{-\tilde{\alpha}}\frac{\langle w\rangle^{\frac{1}{2}}}{|\mathrm{Im}w|} + \kappa^{-1}\left(||F(\cdot)(\kappa H - w)\tilde{\psi}|| + C_6\frac{|w|}{|\mathrm{Im}w|}\right)\right)$$

$$\le C_7\big(t^{-\tilde{\alpha}} + \kappa^{-1}\big)\frac{\langle w\rangle}{|\mathrm{Im}w|},$$

we obtain upon inserting into the right hand side of (2.16)

$$||F\big(t^{-\tilde{\alpha}}x > 1\big)p\tilde{\psi}||^2$$

$$\le C_2 t^{-\mu\tilde{\alpha}}|\mathrm{Im}w|^{-2} + C_3C_7\big(t^{-\tilde{\alpha}} + \kappa^{-1}\big)\frac{\langle w\rangle}{|\mathrm{Im}w|^2}$$

$$\le C_8\big(t^{-\mu\tilde{\alpha}} + \kappa^{-1}\big)\frac{\langle w\rangle}{|\mathrm{Im}w|^2}.$$

□

For our problems we may assume that

(2.17) $\quad I(t,x) \,=\, I(t,0)$ for $|x| \,<\, t^{\breve{\alpha}}$

for any given non–negative $\breve{\alpha} < \mu$ (cf. [E]). In Sections 3 and 4 we may take $\breve{\alpha} = 0$, but in Section 5 we shall need a positive $\breve{\alpha} \,<\, \alpha$.

Under the condition (2.17) for some fixed $\breve{\alpha} \in [0,\alpha)$ we get using Lemma 2.2 (cf. [E]):

Lemma 2.3 *Suppose for this* $\breve{\alpha}$

(2.18) $\quad \mu\breve{\alpha} \,\le\, \delta \,<\, \mu\big(1 + \frac{\breve{\alpha}}{2}\big).$

Then with $\phi^+(t) \,=\, U(t)\phi^+$

(2.19) $\quad \limsup_{t\to+\infty} t^{\frac{1}{2}(\mu+\frac{1}{2}\mu\breve{\alpha}-\delta)}||F\big(|t^{\delta}H| > 1\big)\phi^+(t)|| \,<\, \infty.$

Proof We pick an almost analytic extension $\tilde{F}$ of $F(|\cdot| > 1)$ with

(2.20) $\quad |\Big(\bar{\partial}\tilde{F}\Big)(w)| \,\le\, C_k\langle w\rangle^{-1-k}|\mathrm{Im}w|^k;\; k \,\in\, \mathbf{N},$

yielding the representation (cf. [DG, Appendix C.3])

$$\begin{aligned} &i[H(t), F(|\kappa H| > 1)] \\ &= -\frac{1}{\pi}\int_{\mathbf{C}}\Big(\bar{\partial}\tilde{F}\Big)(w)(\kappa H - w)^{-1}i\Big[I(t), \frac{\kappa}{2}p^2\Big](\kappa H - w)^{-1}dudv; \qquad (2.21) \\ &w = u + iv,\; \kappa \ge 1. \end{aligned}$$

We compute

$$i\Big[I(t),\frac{\kappa}{2}p^2\Big]=\kappa O\Big(t^{-(1+\mu)}\Big)F\Big(\frac{x}{t^{\breve\alpha}}>1\Big)p+h.c.,$$

which combined with (2.21), (2.14) and (2.20) yields to

$$\begin{aligned}&||i[H(t),F(|\kappa H|>1)]||\\ &\le \kappa C_1 t^{-(1+\mu)}\int_{\mathbf{C}}|\Big(\bar\partial\tilde F\Big)(w)|\,|\mathrm{Im}w|^{-2}\langle w\rangle^{\frac12}\Big(t^{-\mu\breve\alpha}+\kappa^{-1}\Big)^{\frac12}dudv \qquad (2.22)\\ &\le \kappa C_2 t^{-(1+\mu)}\Big(t^{-\mu\breve\alpha/2}+\kappa^{-\frac12}\Big).\end{aligned}$$

Since in the state $\phi^+(t)$

$$\langle F(|\kappa H|>1)\rangle_{\phi^+(t)}=-\int_t^\infty ds\langle i[H(s),F(|\kappa H|>1)]\rangle_{\phi^+(s)}ds$$

we obtain from (2.22) that

$$\begin{aligned}\langle F(|\kappa H|>1)\rangle_{\phi^+(t)}&\le\int_t^\infty \kappa C_1 s^{-(1+\mu)}\Big(s^{-\mu\breve\alpha/2}+\kappa^{-\frac12}\Big)ds\\ &=C_2\kappa t^{-(\mu+\mu\breve\alpha/2)}+C_3\kappa^{\frac12}t^{-\mu}.\end{aligned}$$

Upon inserting $\kappa=t^\delta$ with δ obeying (2.18) on the right hand side the resulting expression is $O\Big(t^{\delta-\mu-\mu\breve\alpha/2}\Big)$. □

Lemma 2.4 *Suppose the conditions of Lemma 2.3. Then with the notation* $\phi(t)=U(t)\phi$ *for* $\phi\in\mathcal{H}$

$$\int_1^\infty t^{-1}|\big\langle F\big(|t^\delta H|\approx 1\big)\big\rangle_{\phi(t)}|dt\ \le\ C||\phi||^2. \qquad (2.23)$$

Proof Consider the uniformly bounded family of observables

$$\Phi(t)=F_-\Big(|t^\delta H|\Big);\ t\ge 1.$$

Similarly to (2.21) we can compute using (2.17) and Lemma 2.2 (cf. the proof of Lemma 2.3) the Heisenberg derivative

$$\begin{aligned}&\mathbf{D}\Phi(t):=\frac{d}{dt}\Phi(t)+i[H(t),\Phi(t)]\\ &=-\frac1\pi\int_{\mathbf{C}}\Big(\bar\partial\tilde F\Big)(w)\Big(t^\delta H-w\Big)^{-1}\Big(\delta t^{\delta-1}H+i\Big[I(t),t^\delta\frac12p^2\Big]\Big) \qquad (2.24)\\ &\Big(t^\delta H-w\Big)^{-1}dudv\\ &=\delta t^{-1}t^\delta HG\Big(t^\delta H\Big)+O\Big(t^{-(\mu+1)+\delta-\frac12\mu\breve\alpha}\Big);\ G(E)=\frac{d}{dE}F_-(|E|).\end{aligned}$$

The second term on the right hand side is integrable. Consequently

$$-\int_1^\infty t^{-1}\Big\langle t^\delta HG\Big(t^\delta H\Big)\Big\rangle_{\phi(t)}dt\le C||\phi||^2,$$

from which we obtain the lemma by using the freedom in chosing F_-. □

3 Exterior bound

In this section we don't use (2.2) and (2.3). The notation α_0 is used to denote a number with

(3.1) $\quad \alpha_0 \;>\; \alpha.$

Further constraints will be imposed in the statements that follow. Those are satisfied for $\alpha_0 \approx \alpha$.

Lemma 3.1 *Suppose*

(3.2) $\quad \mu\alpha_0 \;\leq\; \delta \;<\; \big(1+\tfrac{\mu}{2}\big)\alpha_0.$

Then

(3.3) $\quad ||F\big(t^{2\gamma}p^2 < \tfrac{1}{4}\big)F\big(\tfrac{x}{t^{\alpha_0}} > 1\big)F\big(t^{2\gamma}p^2 > 1\big)|| \;=\; O\big(t^{-\infty}\big),$

(3.4) $\quad ||F\big(\tfrac{x}{t^{\alpha_0}} > 1\big)F\big(t^{2\gamma}p^2 > 1\big)F\big(\tfrac{x}{t^{\alpha_0}} < \tfrac{1}{4}\big)|| \;=\; O\big(t^{-\infty}\big),$

(3.5) $\quad ||F\big(\tfrac{x}{t^{\alpha_0}} > 1\big)F\big(t^{2\gamma}p^2 > 1\big)F\big(|t^{\delta}H| < 1\big)|| \;=\; O\big(t^{-\infty}\big).$

Proof The statements (3.3) and (3.4) follow readily from the calculus of pseudodifferential operators, cf. [H, Sections 18.5–6] and [DG, Appendix D.4–5], noticing that

(3.6) $\quad \alpha_0 \;>\; \gamma \;=\; \tfrac{\mu}{2}\alpha.$

(Notice that (3.6) leads to a calculus with "the Planck constant $= t^{\gamma-\alpha_0}$".)

As for (3.5) we suppose

(3.7) $\quad ||F\big(\tfrac{x}{t^{\alpha_0}} > 1\big)F\big(t^{2\gamma}p^2 > 1\big)F\big(|t^{\delta}H| < 1\big)|| \;=\; O\big(t^{-s}\big),$

for some $s \geq 0$ and all such F's. Then we shall show the better bound $O\big(t^{-s-\epsilon}\big)$ with $\epsilon = 2^{-1}\delta - \gamma$ (which is positive by (3.1), (3.2) and (3.6)). For that we compute with $\psi \in \mathcal{H}$ given with $||\psi|| = 1$ and

$$\tilde{\psi} = F_1F_3F_3\psi;$$

$$F_1 = F\Big(\frac{x}{t^{\alpha_0}} > 1\Big),\; F_2 = F\big(t^{2\gamma}p^2 > 1\big),\; F_3 = F\Big(|t^{\delta}H| < 1\Big),$$

using the calculus and (3.4)

$$\begin{aligned}
&\langle 2H\rangle_{\tilde{\psi}} = \Big\langle HF_1^2 + F_1^2H + t^{-2\alpha_0}F_1'^2\Big\rangle_{F_2F_3\psi}\\
&= 2\mathrm{Re}\Big\langle F_1F_2(HF_3)\psi, \tilde{\psi}\Big\rangle + 2\mathrm{Re}\Big\langle O\Big(t^{\gamma-\alpha_0(1+\mu)}\Big)\tilde{F}_1\tilde{F}_2F_3\psi, \tilde{\psi}\Big\rangle\\
&+ t^{-2\alpha_0}||F_1'F_2F_3\psi||^2 + O\big(t^{-\infty}\big)\\
&\leq C\Big(t^{-\delta}||F_1F_2\Big(t^{\delta}HF_3\Big)\psi|| + t^{\gamma-\alpha_0(1+\mu)}||\tilde{F}_1\tilde{F}_2F_3\psi||\Big)\,||\tilde{\psi}||\\
&+ t^{-2\alpha_0}||F_1'F_2F_3\psi||^2 + O\big(t^{-\infty}\big).
\end{aligned} \tag{3.8}$$

Here the last term $O\big(t^{-\infty}\big)$ is estimated uniformly w.r.t. ψ, and $\tilde{F}_1 = \tilde{F}_1\big(\tfrac{x}{t^{\alpha_0}} > 1\big)$ with $\tilde{F}_1(\cdot > 1)$ equal to one on a neighbourhood of the support of $F_1(\cdot > 1)$; and similarly for $\tilde{F}_2$.

On the other hand by (3.3), (3.6) and (3.1)

$$\begin{aligned}
&\langle 2H\rangle_{\tilde{\psi}} = \langle p^2\rangle_{\tilde{\psi}} + \langle 2V\rangle_{\tilde{\psi}}\\
&\geq C_1||\tilde{\psi}||^2t^{-2\gamma} + O\big(t^{-\infty}\big) - C_2||\tilde{\psi}||^2t^{-\mu\alpha_0}\\
&\geq C_3||\tilde{\psi}||^2t^{-2\gamma} + O\big(t^{-\infty}\big).
\end{aligned} \tag{3.9}$$

Combining (3.2), (3.8) and (3.9) gives by reduction

$$
\begin{aligned}
&t^{2\gamma-\delta}C\Big(\Big(||F_1F_2\Big(t^{\delta}HF_3\Big)\psi|| + ||\tilde{F}_1\tilde{F}_2F_3\psi||\Big)||\tilde{\psi}|| + ||F_1'F_2F_3\psi||^2\Big)\\
&\geq ||\tilde{\psi}||^2 + O\big(t^{-\infty}\big),
\end{aligned}
\tag{3.10}
$$

whence by the induction hypothesis (3.7), $||\tilde{\psi}|| \leq t^{-s-\epsilon}C||\psi|| = t^{-s-\epsilon}C$.

□

Lemma 3.2 *Suppose (2.18), (3.1) and (3.2). Then with* $\phi(t) = U(t)\phi$ *for any* $\phi \in \mathcal{H}$

$$
\lim_{t\to+\infty} ||F\big(\tfrac{x}{t^{\alpha_0}} > 1\big)F_-\big(|t^{\delta}H|\big)\phi(t)|| \,=\, 0. \tag{3.11}
$$

Proof By Lemma 3.1

$$
\lim_{t\to+\infty} ||F_+\Big(\frac{x}{t^{\alpha_0}}\Big)F_+(pt^{\gamma})F_-\Big(|t^{\delta}H|\Big)\phi(t)|| = 0,
$$

so we need to show that

$$
\begin{aligned}
&\lim_{t\to+\infty} ||B(t,t^{\alpha_0})\phi(t)|| = 0;\; B(t,\kappa) = F_1F_2F_3,\\
&F_1 = F_+\big(\kappa^{-1}x\big),\; F_2 = F_-(pt^{\gamma}),\; F_3 = F_-\Big(|t^{\delta}H|\Big).
\end{aligned}
\tag{3.12}
$$

For that we compute with $A(t,\kappa) = B(t,\kappa)^*B(t,\kappa)$, $t \geq t_0$ and $\kappa = t^{\alpha_0}$ (the latter considered as fixed)

$$
\langle A(t,\kappa)\rangle_{\phi(t)} \,=\, \langle A(t_0,\kappa)\rangle_{\phi(t_0)} + \int_{t_0}^{t} \langle \mathbf{D}A(s,\kappa)\rangle_{\phi(s)}ds, \tag{3.13}
$$

where the Heisenberg derivative is decomposed as

$$
\begin{aligned}
&\mathbf{D}A(s,\kappa) = T_1 + T_2 + T_3,\\
&T_1 = F_3F_2\big(\mathbf{D}F_1^2\big)F_2F_3,\\
&T_2 = F_3(\mathbf{D}F_2)F_1^2F_2F_3 + h.c.,\\
&T_3 = (\mathbf{D}F_3)F_2F_1^2F_2F_3 + h.c..
\end{aligned}
\tag{3.14}
$$

To handle the contribution from the term T_1 we estimate (using the calculus of pseudodifferential operators to handle the double commutator)

$$
\begin{aligned}
&F_2\big(\mathbf{D}F_1^2\big)F_2 = 2^{-1}\kappa^{-1}F_2\big(pG^2\big(\kappa^{-1}x\big) + G^2\big(\kappa^{-1}x\big)p\big)F_2\\
&= \kappa^{-1}s^{-\gamma}F_2\big\{G\big(\kappa^{-1}x\big)H(s^{\gamma}p)G\big(\kappa^{-1}x\big) + 2^{-1}\big[\big[H(s^{\gamma}p),G\big(\kappa^{-1}x\big)\big],G\big(\kappa^{-1}x\big)\big]\big\}F_2\\
&\leq C\kappa^{-1}s^{-\gamma}\big(1+\kappa^{-2}s^{2\gamma}\big);\\
&G(x) = \big(F_+^{2\prime}\big)^{\frac{1}{2}}(x),\; H(\xi) = \xi F_-\big(4^{-1}\xi\big).
\end{aligned}
\tag{3.15}
$$

Upon inserting (3.15) into the integrand yields to

$$
\int_{t_0}^{t} \langle T_1\rangle_{\phi(s)}ds \,\leq\, C\kappa^{-1}\big(t^{1-\gamma} + \kappa^{-2}t^{1+\gamma}\big). \tag{3.16}
$$

As for the contribution from the term T_2 we notice (cf. (2.24))

$$
\begin{aligned}
\mathbf{D}F_2 &= i[V, F_2] \\
&- \frac{1}{\pi}\int\limits_{\mathbf{C}} \left(\bar{\partial}\tilde{F}_-\right)(w)(s^\gamma p - w)^{-1}\left(\gamma s^{\gamma-1}p + i[I(s), s^\gamma p]\right)(s^\gamma p - w)^{-1}dudv \\
&= T + \gamma s^{-1}H(s^\gamma p) + O\left(s^{\gamma-(\mu+1)}\right); \\
T &= i[V, F_2],\ H(\xi) = \xi\frac{d}{d\xi}F_-(\xi).
\end{aligned} \tag{3.17}
$$

Since the form of the function H in (3.17) is $H(\xi) = F\left(\xi^2 > 1\right)$ and obviously

$$F_1 \;=\; F\left(s^{-\alpha_0}x > 1\right)F_1 \tag{3.18}$$

for some function $F(\cdot > 1)$, we can use Lemma 3.1 to conclude that

$$F_3\gamma s^{-1}H(s^\gamma p)F_1^2F_2F_3 + h.c. \;=\; O\left(s^{-\infty}\right). \tag{3.19}$$

As for the term T in (3.17) we use (3.18) and Lemma 3.1 again, and the calculus of pseudodifferential operators, to write

$$F_3TF_1^2F_2F_3 + h.c. \;=\; O\left(s^{-\infty}\right). \tag{3.20}$$

Using finally for the third term in (3.17) that $\gamma - \mu - 1 < -1$ we conclude that

$$\int\limits_{t_0}^{t}\langle T_2\rangle_{\phi(s)}ds \;\le\; \int\limits_{t_0}^{\infty}\langle T_2\rangle_{\phi(s)}ds \;\le\; Ct_0^{-\epsilon} \text{ for some } \epsilon > 0. \tag{3.21}$$

For the contribution from the term T_3 we use (2.24) and (2.23). The contribution from the error term in (2.24) is treated like T_2 in (3.21), while the first term on the right hand side of (2.24) needs to be symmetrized before (2.23) can be applied: We write $t^\delta HG\left(t^\delta H\right)F_3 = -\tilde{G}^2\left(t^\delta H\right)$ and

$$t^{-1}t^\delta HG\left(t^\delta H\right)F_2F_1^2F_2F_3 \;=\; -t^{-1}\tilde{G}\left(t^\delta H\right)F_2F_1^2F_2\tilde{G}\left(t^\delta H\right) + R, \tag{3.22}$$

where by commutation (cf. (2.24) and Lemma 2.2)

$$R \;=\; O\left(s^{\delta+\gamma-(\mu+1)\alpha_0-1}\right) + \kappa^{-1}O\left(s^{\delta-\frac{\mu}{2}\alpha_0-1}\right). \tag{3.23}$$

The contribution from the first term on the right hand side of (3.22) is by (2.23) of the form

$$o\left(t_0^0\right) \text{ for } t_0 \to \infty,$$

uniformly in $t \;\ge\; t_0$.

As for the the term R we use that $\delta + \gamma - (\mu+1)\alpha_0 - 1 < -1$ to treat the first term in (3.23). The second is estimated like

$$\int\limits_{t_0}^{t}\kappa^{-1}O\left(s^{\delta-\frac{\mu}{2}\alpha_0-1}\right)ds \;\le\; C\kappa^{-1}t^{\delta-\frac{\mu}{2}\alpha_0}. \tag{3.24}$$

Now using $\kappa = t^{\alpha_0}$ in (3.16) and (3.24) we finally conclude from (3.13) that

$$\langle A(t, t^{\alpha_0})\rangle_{\phi(t)} \;\le\; \langle A(t_0, t^{\alpha_0})\rangle_{\phi(t_0)} + o\left(t_0^0\right) + o\left(t^0\right), \tag{3.25}$$

where the decay of the middle term is uniform w.r.t. to $t \geq t_0$.

Obviously we get from (3.25) that

$$(3.26) \quad \langle A(t, t^{\alpha_0}) \rangle_{\phi(t)} \, = \, o\big(t^0\big)$$

by first chosing a large t_0 to bound the middle term and noticing that indeed the first term has the form of the last term (for the fixed t_0).

We have proved (3.12). □

Clearly by combining Lemma 2.3 and Lemma 3.2 we obtain:

Corollary 3.3 *Under the conditions of Lemma 3.2 and with $\phi^+(t)$ given as in Lemma 2.3*

$$(3.27) \quad \lim_{t \to +\infty} ||\phi^+(t) \, - \, F_-\big(\tfrac{x}{t^{\alpha_0}}\big) F_-\big(|t^\delta H|\big)\phi^+(t)|| \, = \, 0.$$

Lemma 3.4 *Under the conditions of Lemma 3.2 and with the notation $\phi(t) = U(t)\phi$ for $\phi \in \mathcal{H}$,*

$$(3.28) \quad \int_1^\infty t^{-1} |\big\langle F\big(\tfrac{x}{t^{\alpha_0}} \approx 1\big)\big\rangle_{F_-(|t^\delta H|)\phi(t)}| dt \, \leq \, C||\phi||^2.$$

Proof Consider the uniformly bounded family of observables

$$\Phi(t) = F_- F_+\Big(\frac{x}{t^{\alpha_0}}\Big) F_-; \; F_- = F_-\Big(|t^\delta H|\Big).$$

As in the proof of Lemma 2.4 we compute the Heisenberg derivative. The contribution from the two factors F_- is treated by (2.24) and (2.23) (by symmetrizing cf. (3.22) and (3.23)). So we need only to consider the contribution from

$$(3.29) \quad \mathbf{D} F_+\big(\tfrac{x}{t^{\alpha_0}} > 1\big) \, = \, \tfrac{1}{2}\big(\tfrac{p}{t^{\alpha_0}} - \alpha_0 \tfrac{x}{t^{\alpha_0+1}}\big) F'_+\big(\tfrac{x}{t^{\alpha_0}}\big) \, + \, F'_+\big(\tfrac{x}{t^{\alpha_0}}\big) \tfrac{1}{2}\big(\tfrac{p}{t^{\alpha_0}} - \alpha_0 \tfrac{x}{t^{\alpha_0+1}}\big).$$

For each term on the right hand side we insert

$$(3.30) \quad I \, = \, F_+\big(t^{2\gamma} p^2\big) \, + \, F_-\big(t^{2\gamma} p^2\big)$$

to the left and to the right, respectively. The first term in (3.30) contributes by Lemma 3.1 by a term of the form $O\big(t^{-\infty}\big)$. Moreover

$$(3.31) \quad \begin{aligned} &\frac{1}{2} F_-\big(t^{2\gamma} p^2\big) \Big(\frac{p}{t^{\alpha_0}} - \alpha_0 \frac{x}{t^{\alpha_0+1}}\Big) F'_+\Big(\frac{x}{t^{\alpha_0}}\Big) + h.c. \\ &\leq -\alpha_0 F_-^{\frac{1}{2}}(\cdot) \frac{x}{t^{\alpha_0+1}} F'_+\Big(\frac{x}{t^{\alpha_0}}\Big) F_-^{\frac{1}{2}}(\cdot) + C_1 t^{-\gamma-\alpha_0} + C_2 t^{-1+\gamma-\alpha_0}. \end{aligned}$$

Since $-\gamma - \alpha_0 < -1$ and $\gamma - \alpha_0 < 0$ we obtain (3.28) by integrating (3.31) and using Lemma 3.1 again. □

4 Intermediate bound

In this and the next section we impose (2.2). We shall consider two decreasing sequences of positive numbers $(\gamma_j)_1^\infty$ and $(\alpha_j)_0^\infty$ assumed to satisfy:

$$(4.1) \quad \delta \, < \, \alpha_j \, + \, \gamma_j.$$

$$(4.2) \quad \delta + \gamma_j \, < \, (\mu + 1)\alpha_j.$$

(4.3) $\quad \mu > \gamma_j > \frac{\mu}{2}\alpha_{j-1}.$

(4.4) $\quad \delta > \mu\alpha_{j-1}.$

(4.5) $\quad \alpha_j + \gamma_j < 1.$

Here α_0 and δ are chosen in agreement with the conditions (2.18), (3.1) and (3.2).

We shall be interested in the "limiting regions": $\alpha_0 \approx \alpha$, $\delta \approx \mu\alpha$, $\gamma_j \approx \frac{\mu}{2}\alpha_{j-1}$ and $\alpha_j \approx \frac{\delta+\gamma_j}{\mu+1}$. (See the proof of Lemma 4.6 below for precise requirements.)

We have the following analogue of Lemma 3.1.

Lemma 4.1 *Suppose (4.1)–(4.4). Then*

(4.6) $$||t^{2\gamma_j}p^2F\big(t^{2\gamma_j}p^2>4\big)F\big(\tfrac{x}{t^{\alpha_j}}>1, \tfrac{x}{t^{\alpha_{j-1}}}<1\big)F\big(t^{2\gamma_j}p^2<1\big)|| = O\big(t^{-\infty}\big),$$

(4.7) $$||F\big(\tfrac{x}{t^{\alpha_j}}>1\big)F\big(t^{2\gamma_j}p^2<1\big)F\big(\tfrac{x}{t^{\alpha_j}}<\tfrac{1}{4}\big)|| = O\big(t^{-\infty}\big),$$

(4.8) $$||F\big(\tfrac{x}{t^{\alpha_j}}>1, \tfrac{x}{t^{\alpha_{j-1}}}<1\big)F\big(t^{2\gamma_j}p^2<1\big)F\big(|t^{\delta}H|<1\big)|| = O\big(t^{-\infty}\big).$$

Proof By the assumptions

$$\mu\alpha_j+\gamma_j < \mu\alpha_{j-1}+\gamma_j < \delta+\gamma_j < \alpha_j(\mu+1),$$

and therefore by subtraction

(4.9) $\quad \gamma_j < \alpha_j.$

Similarly to the proof of (3.3) and (3.4) we can use (4.9) to conclude (4.6) and (4.7).

As for (4.8) we can proceed as in the proof of (3.5):

Define

$$\tilde\psi = F\Big(\frac{x}{t^{\alpha_j}}>1, \frac{x}{t^{\alpha_{j-1}}}<1\Big)F\big(t^{2\gamma_j}p^2<1\big)F\Big(|t^{\delta}H|<1\Big)\psi = F_1F_2F_3\psi.$$

Then

$$\begin{aligned}
&-\langle 2H\rangle_{\tilde\psi} = \Big\langle HF_1^2+F_1^2H+t^{-2\alpha_j}F_1'^2\Big\rangle_{F_2F_3\psi}\\
(4.10)\quad &\le C\Big(t^{-\delta}||F_1F_2\Big(t^{\delta}HF_3\Big)\psi|| + t^{\gamma_j-\alpha_j(1+\mu)}||\tilde F_1\tilde F_2F_3\psi||\Big)\,||\tilde\psi||\\
&+t^{-2\alpha_j}||F_1'F_2F_3\psi||^2+O\big(t^{-\infty}\big).
\end{aligned}$$

On the other hand by (4.6) and (4.3)

$$\begin{aligned}
&\langle -2H\rangle_{\tilde\psi} = \langle -2V\rangle_{\tilde\psi} - \big\langle p^2\big\rangle_{\tilde\psi}\\
(4.11)\quad &\ge C_1||\tilde\psi||^2t^{-\mu\alpha_{j-1}} - C_2||\tilde\psi||^2t^{-2\gamma_j}+O\big(t^{-\infty}\big)\\
&\ge C_3||\tilde\psi||^2t^{-\mu\alpha_{j-1}}+O\big(t^{-\infty}\big).
\end{aligned}$$

Combining (4.10), (4.11) and (4.4) gives the result by induction, cf. the proof of Lemma 3.1. □

Lemma 4.2 *Under the conditions (2.18), (3.1), (3.2) and (4.1)–(4.5), for any $\phi(t) = U(t)\phi$ and $j \ge 1$*

$H(1,j)$ $$\int_1^\infty t^{-\alpha_j}\big\langle pF_+^2(t^{\gamma_j}p)\big\rangle_{\tilde\phi_j(t)}dt \le C||\phi||^2,$$

$$H(2,j) \qquad \int_1^\infty t^{-\alpha_j}\big\langle -pF_+^2(-t^{\gamma_j}p)\big\rangle_{\tilde\phi_j(t)}dt \le C||\phi||^2,$$

where

$$\tilde\phi_j(t) = B_j(t)\phi(t);\ B_j(t) = \big(F_+^{2\prime}\big)^{\frac12}\Big(\frac{x}{t^{\alpha_j}}\Big)F_-\Big(|t^\delta H|\Big).$$

Proof Let

$$(4.12)\qquad \begin{aligned}&\Phi_j(t) = F_4F_3F_2F_1^2F_2F_3F_4;\\ &F_1 = F_+\Big(\frac{x}{t^{\alpha_j}}\Big),\ F_2 = F_-\Big(\frac{x}{t^{\alpha_{j-1}}}\Big),\ F_3 = F_+(t^{\gamma_j}p),\ F_4 = F_-\Big(|t^\delta H|\Big).\end{aligned}$$

We start by showing $H(1,1)$:

Consider the uniformly bounded family of observables Φ_1. Its Heisenberg derivative is decomposed as

$$(4.13)\qquad \begin{aligned}&\mathbf{D}\Phi_1(t) = T_1+T_2+T_3+T_4;\\ &\quad T_1 = F_4F_3F_2\mathbf{D}\big(F_1^2\big)F_2F_3F_4,\\ &\quad T_2 = F_4F_3(\mathbf{D}F_2)F_1^2F_2F_3F_4 + h.c.,\\ &\quad T_3 = F_4(\mathbf{D}F_3)F_2F_1^2F_2F_3F_4 + h.c.,\\ &\quad T_4 = (\mathbf{D}F_4)F_3F_2F_1^2F_2F_3F_4 + h.c..\end{aligned}$$

By Lemmas 3.4, 4.1 and 2.4, respectively, and commutation (the latter yielding integrable errors)

$$(4.14)\qquad \int_1^\infty |\langle T_k\rangle_{\phi(t)}|dt \le C||\phi||^2;\ k = 2,3,4.$$

Notice that we also use the proof of Lemma 3.4 to treat the term T_2. Notice for treating the term T_4 that we use (2.24). The first term on the right hand side is symmetrized with an integrable error. This can be shown by combining Lemma 2.2 and the fact that

$$(4.15)\qquad \delta < \big(1+\tfrac{\mu}{2}\big)\alpha_j,$$

cf. (4.1) and (4.2).

As for the term T_1 we compute, cf. (3.29),

$$(4.16)\qquad \begin{aligned}&\mathbf{D}F_1^2 = \frac12\Big(\frac{p}{t^{\alpha_1}} - \alpha_1\frac{x}{t^{\alpha_1+1}}\Big)F_+^{2\prime}\Big(\frac{x}{t^{\alpha_1}}\Big) + h.c.\\ &= \big(F_+^{2\prime}\big)^{\frac12}\Big(\frac{x}{t^{\alpha_1}}\Big)\Big(\frac{p}{t^{\alpha_1}} - \alpha_1\frac{x}{t^{\alpha_1+1}}\Big)\big(F_+^{2\prime}\big)^{\frac12}\Big(\frac{x}{t^{\alpha_1}}\Big).\end{aligned}$$

By commutation under use of the calculus of pseudodifferential operators, Lemma 4.1, (4.5) and (4.9)

$$(4.17)\qquad \begin{aligned}&T_1 = B_1(t)^\star F_3\Big(\frac{p}{t^{\alpha_1}} - \alpha_1\frac{x}{t^{\alpha_1+1}}F_-\Big(4^{-1}\frac{x}{t^{\alpha_1}}\Big)\Big)F_3B_1(t) + O\big(t^{-\infty}\big)\\ &\ge \frac12 B_1(t)^\star\frac{p}{t^{\alpha_1}}F_3^2B_1(t) + O\big(t^{-\infty}\big).\end{aligned}$$

From (4.17) it follows that

$$(4.18)\qquad \int_1^T 2^{-1}t^{-\alpha_1}\big\langle pF_+^2(t^{\gamma_1}p)\big\rangle_{\tilde\phi_1(t)}dt \le \int_1^T \langle T_1\rangle_{\phi(t)}dt + C||\phi||^2$$

for C independent of $T>1$.

Combining (4.14) and (4.18) yields to $H(1,1)$.

Suppose now $H(1,j)$. We need to show $H(1,j+1)$:

We proceed by considering $\Phi_{j+1}(t)$ given in (4.12). Its Heisenberg derivative is decomposed as in (4.13). We claim that the analogue of (4.14) holds true: For the term T_2 we use $H(1,j)$ and the fact that $\gamma_{j+1}<\gamma_j$. (The latter allows for inserting $F_+(t^{\gamma_j}p)$.) For the terms T_3 and T_4 we use Lemma 4.1, and Lemma 2.4 and (4.15), respectively.

By the same arguments used for (4.17) we get a complete analogue statement and therefore also the following analogue of (4.18)

$$\int_1^T 2^{-1}t^{-\alpha_{j+1}}\big\langle pF_+^2(t^{\gamma_{j+1}}p)\big\rangle_{\tilde\phi_{j+1}(t)}dt \le \int_1^T \langle T_1\rangle_{\phi(t)}dt + C||\phi||^2, \tag{4.19}$$

yielding $H(1,j+1)$.

Next we show $H(2,1)$:

We consider

$$\begin{aligned}&\Phi(t)=F_4F_3F_2F_1^2F_2F_3F_4;\\&F_1=F_+\Big(\frac{x}{t^{\alpha_1}}\Big),\ F_2=F_-\Big(\frac{x}{t^{\alpha_0}}\Big),\ F_3=F_+(-t^{\gamma_1}p),\ F_4=F_-\Big(|t^\delta H|\Big).\end{aligned}\tag{4.20}$$

Commuting as in (4.13) and arguing as before we obtain the complete analogue of (4.14). The analogue of (4.17) is

$$\begin{aligned}&T_1=B_1(t)^*F_3\Big(\frac{p}{t^{\alpha_1}}-\alpha_1\frac{x}{t^{\alpha_1+1}}F_-\Big(4^{-1}\frac{x}{t^{\alpha_1}}\Big)\Big)F_3B_1(t)\\&\ +O\Big(t^{(\gamma_1-\alpha_1)N}\Big)+O\big(t^{-\infty}\big)\\&\le B_1(t)^\star\frac{p}{t^{\alpha_1}}F_3^2B_1(t)+O\big(t^{-\infty}\big).\end{aligned}\tag{4.21}$$

Upon integrating (4.17)

$$\int_1^T t^{-\alpha_1}\big\langle -pF_+^2(t^{\gamma_1}p)\big\rangle_{\tilde\phi_1(t)}dt \le -\int_1^T\langle T_1\rangle_{\phi(t)}dt + C||\phi||^2. \tag{4.22}$$

Clearly $H(2,1)$ follows from (4.22) and the analogue of (4.14).

The verification of $H(2,j+1)$ given $H(2,j)$ follows the same pattern using the hypothesis $H(2,j)$, and for the term T_1 a treatment similar to (4.21). $\square$

Lemma 4.3 *Under the conditions of Lemma 4.3*

$$\lim_{t\to+\infty}||F\big(\tfrac{x}{t^{\alpha_j}}>1,\tfrac{x}{t^{\alpha_{j-1}}}<1\big)F_-(t^{\gamma_j}p)F_-\big(|t^\delta H|\big)\phi(t)||=0. \tag{4.23}$$

Proof By Lemma 4.1 it suffices to show that

$$\lim_{t\to+\infty}||F\big(\tfrac{x}{t^{\alpha_j}}>1,\tfrac{x}{t^{\alpha_{j-1}}}<1\big)F_+(-t^{\gamma_j}p)F_-\big(|t^\delta H|\big)\phi(t)||=0. \tag{4.24}$$

We prove (4.24) by mimicking the proof of Lemma 3.2:

We introduce for $\kappa>1$

$$\begin{aligned}&B(t,\kappa)=F_1F_2F_3F_4,\ A(t,\kappa)=B(t,\kappa)^*B(t,\kappa);\\&F_1=F_+\big(\kappa^{-1}x\big),\ F_2=F_-\Big(\frac{x}{t^{\alpha_{j-1}}}\Big),\ F_3=F_+(-t^{\gamma_j}p),\ F_4=F_-\Big(|t^\delta H|\Big).\end{aligned}\tag{4.25}$$

Then (3.13) holds with

$$\begin{aligned}
&\mathbf{D}A(t,\kappa) = T_1 + T_2 + T_3 + T_4;\\
&T_1 = F_4F_3F_2\mathbf{D}\big(F_1^2\big)F_2F_3F_4,\\
(4.26)\quad &T_2 = F_4F_3(\mathbf{D}F_2)F_1^2F_2F_3F_4 + h.c.,\\
&T_3 = F_4(\mathbf{D}F_3)F_2F_1^2F_2F_3F_4 + h.c.,\\
&T_4 = (\mathbf{D}F_4)F_3F_2F_1^2F_2F_3F_4 + h.c..
\end{aligned}$$

We let $\kappa = t^{\alpha_j}$ and need to show the analogue of (3.25). The contributions to the integral from the terms T_3 and T_4 are handled using Lemmas 4.1 and 2.4, respectively, and the one from T_2 using $H(2, j-1)$ of Lemma 4.2 (for $j > 1$) or Lemma 3.4 (for $j = 1$), cf. the proof of Lemma 4.2.

As for T_1 the proof of (4.17) shows that

$$\begin{aligned}
&T_1 = B_j(t,s)^*F_+(-s^{\gamma_j}p)t^{-\alpha_j}pF_+(-s^{\gamma_j}p)B_j(t,s)\\
(4.27)\quad &+O\Big(s^{(\gamma_j-\alpha_j)N}\Big) + O\big(s^{-\infty}\big)\\
&\le O\big(s^{-\infty}\big),
\end{aligned}$$

where we have used the notation

$$B_j(t,s) = \big(F_+^{2\prime}\big)^{\frac{1}{2}}\Big(\frac{x}{t^{\alpha_j}}\Big)F_-\Big(\frac{x}{s^{\alpha_{j-1}}}\Big)F_-\Big(|s^{\delta}H|\Big),\ t \ge s \ge 1.$$

Clearly (4.27) implies the estimate

$$(4.28)\quad \int_{t_0}^{t}\langle T_1\rangle_{\phi(s)}ds \ \le\ O\big(t_0^{-\infty}\big).$$

The above treatment of the the terms T_2, T_3 and T_4 combined with (4.28) yields to the analogue of (3.25) and therefore to (4.24). □

Lemma 4.4 *Under the conditions of Lemma 4.2 there exists*

$$(4.29)\quad P_j^+\phi = \lim_{t\to+\infty} U(t)^*F_-\big(|t^{\delta}H|\big)F_+^2\big(\tfrac{x}{t^{\alpha_j}}\big)F_-^2\big(\tfrac{x}{t^{\alpha_{j-1}}}\big)F_-\big(|t^{\delta}H|\big)\phi(t).$$

Proof By Lemma 4.3 it suffices to show the existence of

$$\lim_{t\to+\infty} U(t)^*\Phi_j(t)\phi(t),$$

where $\Phi_j(t)$ is given in (4.12). But by using $H(1,j)$ and $H(1,j-1)$ (the latter for $j > 1$; otherwise by Lemma 3.4) as well as the proof of these statements yields to this existence result. □

Lemma 4.5 *Under the conditions of Lemma 4.2*

$$(4.30)\quad P_j^+ \ =\ 0;\ j\ \ge\ 2.$$

Proof Define

$$(4.31)\quad K(x) \ =\ \begin{cases} \displaystyle\int_R^x (-2V(y))^{-\frac{1}{2}}dy;\ x \ge R,\\ 0;\ x < R,\end{cases}$$

with R given by the assumption (2.2). Obviously

$$K(x) \le C|x|^{1+\frac{\mu}{2}}. \tag{4.32}$$

Let for $\kappa > 1$

$$\begin{aligned} &A(t,\kappa) = B(t,\kappa)^\star B(t,\kappa),\ B(t,\kappa) = F_0F_1F_2F_3F_4;\\ &F_0 = F_+\big(\kappa^{-1}|t-K(x)|\big),\\ &F_1 = F_+\Big(\frac{x}{t^{\alpha_j}}\Big),\ F_2 = F_-\Big(\frac{x}{t^{\alpha_{j-1}}}\Big),\ F_3 = F_+(t^{\gamma_j}p),\ F_4 = F_-\Big(|t^\delta H|\Big). \end{aligned} \tag{4.33}$$

We pick $\beta_j \in (0,1)$ such that

$$\beta_j - \mu\alpha_{j-1} + \min\big(\delta, (\mu+1)\alpha_j - \tfrac{\mu}{2}\alpha_{j-1}\big) > 1. \tag{4.34}$$

Notice that adding (4.3) and (4.4) in conjunction with (4.2) yields to

$$\tfrac{3}{2}\mu\alpha_{j-1} < (\mu+1)\alpha_j, \tag{4.35}$$

and also that by (4.4), $\mu\alpha_{j-1} < \delta$.

We shall again follow the scheme of the proof of Lemma 3.2. We use (3.13) for the above $A(t,\kappa)$ with $\kappa = t^{\beta_j}$.

The Heisenberg derivative is decomposed into five terms. Only the contribution from the term

$$T_0 = S^*\big(\mathbf{D}F_0^2\big)S;\ S = F_1F_2F_3F_4, \tag{4.36}$$

needs careful consideration. The remaining terms are treated by methods used before in this section. We notice though that symmetrizing the contribution from the first term on the right hand side of (2.24) in treating the Heisenberg derivative of F_4 leads to the requirement

$$\delta < \beta_j - \tfrac{\mu}{2}(\alpha_{j-1} - \alpha_j), \tag{4.37}$$

(to commute through F_0).

The condition (4.37) is fulfilled for β_j close to one since by (4.3), (4.5) and (4.15) (used in the indicated order)

$$1 - \frac{\mu}{2}(\alpha_{j-1} - \alpha_j) > 1 - \gamma_j + \frac{\mu}{2}\alpha_j > \Big(1 + \frac{\mu}{2}\Big)\alpha_j > \delta.$$

We compute

$$\begin{aligned} &\mathbf{D}F_0^2 = \frac{\kappa^{-1}}{2}\Big\{\Big(I - p(-2V)^{-\frac{1}{2}}\Big)\tilde F + h.c.\Big\};\\ &\tilde F = F\big(\kappa^{-1}|t-K(x)|\big),\ F(s) = \frac{d}{ds}F_+^2(|s|). \end{aligned} \tag{4.38}$$

Moreover, putting

$$\begin{aligned} &G = pF_+(4t^{\gamma_j}p)\\ &\ + (-2V)^{\frac{1}{2}}F_+\Big(4\frac{x}{t^{\alpha_j}}\Big)F_-\Big(4^{-1}\frac{x}{t^{\alpha_{j-1}}}\Big) + I - F_+\Big(4\frac{x}{t^{\alpha_j}}\Big)F_-\Big(4^{-1}\frac{x}{t^{\alpha_{j-1}}}\Big), \end{aligned}$$

by a commutation (cf. the proof of (4.6))

$$
\begin{aligned}
& S^*\Big(I - p(-2V)^{-\frac{1}{2}}\Big)\tilde{F}S \\
& = -S^*\Big(p - (-2V)^{\frac{1}{2}}\Big)GG^{-1}(-2V)^{-\frac{1}{2}}\tilde{F}S \\
& = -S^*\Big(p - (-2V)^{\frac{1}{2}}\Big)\Big(p + (-2V)^{\frac{1}{2}}\Big)G^{-1}(-2V)^{-\frac{1}{2}}\tilde{F}S + O\big(t^{-\infty}\big) \\
& = -S^*\Big(2H - i(-2V)^{-\frac{1}{2}}V'\Big)G^{-1}\tilde{F}(-2V)^{-\frac{1}{2}}S + O\big(t^{-\infty}\big).
\end{aligned} \tag{4.39}
$$

But

$$
\begin{aligned}
& ||(-2V)^{-\frac{1}{2}}S|| = O\Big(t^{\frac{\mu}{2}\alpha_{j-1}}\Big), \\
& ||G^{-1}|| = O\Big(t^{\frac{\mu}{2}\alpha_{j-1}}\Big), \\
& ||S^*(-2V)^{-\frac{1}{2}}V'|| = O\Big(t^{\frac{\mu}{2}\alpha_{j-1}-(\mu+1)\alpha_j}\Big).
\end{aligned} \tag{4.40}
$$

Combining (4.38)–(4.40) and Lemmas 2.2, 4.1 and 4.2 (the lemmas to commute H in (4.39) to the left) yields to

$$\int_{t_0}^{t} \langle T_0\rangle_{\phi(s)} ds \,=\, O\Big(t^{1-\beta_j+\mu\alpha_{j-1}-\min\left(\delta,(\mu+1)\alpha_j-\frac{\mu}{2}\alpha_{j-1}\right)}\Big), \tag{4.41}$$

which by (4.34) is $o\big(t^0\big)$.

In conclusion

$$||B\Big(t, t^{\beta_j}\Big)\phi(t)|| = o\big(t^0\big).$$

Moreover since by (4.5) and (4.3)

$$\alpha_j \,\le\, \alpha_1 \,<\, 1 - \tfrac{\mu}{2}\alpha_0 \,<\, \alpha, \tag{4.42}$$

we readily obtain in combination with (4.32) that for $j \ge 2$

$$F_-\Big(t^{-\beta_j}|t - K(x)|\Big)F_2 = 0 \text{ for large } t.$$

Putting together it follows that for $j \ge 2$

$$\Big\langle P_j^+\Big\rangle_\phi = \lim_{t\to+\infty} ||S\phi(t)||^2 = 0.$$

□

We can now prove (2.12).

Proposition 4.6 *For any* $\epsilon > 0$

$$\lim_{t\to+\infty} ||F\big(\tfrac{x}{t^{\alpha_{cr}+\epsilon}} > 1, \tfrac{x}{t^{\alpha-\epsilon}} < 1)\big)\phi^+(t)|| \,=\, 0. \tag{4.43}$$

Proof We shall use Lemmas 2.3 and 4.5 using the freedom in chosing our parameters obeying (2.18), (3.1), (3.2) and (4.1)–(4.5):

As for the latter we define for sufficiently small $\sigma > 0$:

$$
\begin{aligned}
&\alpha_0 = \alpha + \frac{\sigma}{\mu},\\
&\delta = \mu\alpha + 2\sigma,\\
(4.44)\qquad &\gamma_j = \frac{\mu}{2}\alpha_{j-1} + \sigma,\\
&\alpha_1 = 1 - \gamma_1 - \sigma,\\
&\alpha_j = (\mu+1)^{-1}(\delta + \gamma_j + \sigma);\ j \geq 2.
\end{aligned}
$$

Then clearly the relations (2.18), (3.1) and (3.2) hold. Clearly (4.4) holds for $j = 0$ and hence in general (provided $(\alpha_j)_0^\infty$ is decreasing, see below). By definition the relation (4.3) is true. Obviously (4.1) and (4.5) hold for $j = 1$, and (4.2) for $j \geq 2$. Moreover

$$(4.45)\qquad (\mu+1)^{-1}(\delta + \gamma_1 + \sigma) < \alpha_1$$

(again for small σ). Consequently (4.2) holds for $j = 1$ as well. We also have that $\alpha_1 < \alpha$.

For $j \geq 2$

$$(4.46)\qquad \alpha_j = (\mu+1)^{-1}\big(\delta + \tfrac{\mu}{2}\alpha_{j-1} + 2\sigma\big),$$

which yields to the existence of $\alpha_\infty = \lim_{n\to\infty} \alpha_n$. Upon substituting in (4.46) we obtain

$$\alpha_\infty = (\mu+1)^{-1}\Big(\delta + \frac{\mu}{2}\alpha_\infty + 2\sigma\Big),$$

which is solved by

$$\alpha_\infty = \alpha(\mu\alpha + 4\sigma).$$

Since by (4.45)

$$
\begin{aligned}
&\alpha_2 = (\mu+1)^{-1}\Big(\delta + \frac{\mu}{2}\alpha_1 + 2\sigma\Big)\\
&< (\mu+1)^{-1}\Big(\delta + \frac{\mu}{2}\alpha_0 + 2\sigma\Big) = (\mu+1)^{-1}(\delta + \gamma_1 + \sigma) < \alpha_1
\end{aligned}
$$

it follows that indeed the sequence $(\alpha_j)_0^\infty$ is decreasing. Then by the definition the same holds for $(\gamma_j)_1^\infty$. In particular (4.5) hold for $j \geq 2$ as well.

It remains to show (4.1) for $j \geq 2$: But

$$\alpha_j + \gamma_j > \Big(1 + \frac{\mu}{2}\Big)\alpha_\infty + \sigma > \mu\alpha + 5\sigma > \delta.$$

Given $\epsilon > 0$ we fix $\sigma > 0$ so small that in addition to the above requirements

$$(4.47)\qquad \alpha_{cr} + \epsilon > \alpha_\infty \text{ and } \alpha - \epsilon \leq \alpha_1.$$

Then for $j = J$ large enough

$$(4.48)\qquad \alpha_{cr} + \epsilon \geq \alpha_J.$$

Now using Lemma 2.3, (4.47) and (4.48) the proposition obviously follows if we can prove

$$\lim_{t\to+\infty} ||F_+\Big(\frac{x}{t^{\alpha_J}}\Big)F_-\Big(\frac{x}{t^{\alpha_1}}\Big)F_-\Big(t^\delta|H|\Big)\phi^+(t)|| = 0.$$

For that we insert repeatedly

$$I = F_+\Big(\frac{x}{t^{\alpha_j}}\Big) + F_-\Big(\frac{x}{t^{\alpha_j}}\Big);\ 2 \le j \le J-1,$$

to the left and invoke Lemma 4.5. □

For a similar application in Section 6 we state the following result which readily follows from the methods of this section.

Lemma 4.7 *Under the conditions of Lemma 4.2 there exists for* $j \ge 1$

(4.49) $$Q_j^+\phi = \lim_{t\to+\infty} U(t)^* F_-\big(|t^\delta H|\big)F_-^2\big(\tfrac{x}{t^{\alpha_j}}\big)F_-\big(|t^\delta H|\big)\phi(t).$$

Proof Notice the following consequence of Lemmas 4.2 and 4.1:

(4.50) $$\int_1^\infty t^{-\alpha_j-\gamma_j}\big\langle\big(F_+^{2\prime}\big)\big(\tfrac{x}{t^{\alpha_j}}\big)\big\rangle_{F_-(|t^\delta H|)\phi}dt \le C||\phi||^2.$$

The contribution to the Heisenberg derivative from each of the factors $F_-\big(|t^\delta H|\big)$ is treated by Lemma 2.4.

We compute

(4.51) $$\mathbf{D}F_-^2\big(\tfrac{x}{t^{\alpha_j}}\big) = \tfrac{1}{2}\Big(\tfrac{p}{t^{\alpha_j}} - \alpha_1\tfrac{x}{t^{\alpha_j+1}}\Big)F_-^{2\prime}\big(\tfrac{x}{t^{\alpha_j}}\big) + h.c.$$

Using (4.5) and (4.50) it remains to treat the contribution

$$F_-\Big(|t^\delta H|\Big)\bigg(\frac{1}{2}\frac{p}{t^{\alpha_j}}F_-^{2\prime}\Big(\frac{x}{t^{\alpha_j}}\Big) + h.c.\bigg)F_-\Big(|t^\delta H|\Big).$$

For that we substitute

$$I = F_+(t^{\gamma_j}p) + F_-\big(t^{2\gamma_j}p^2\big) + F_+(-t^{\gamma_j}p).$$

The middle term and the other terms contribute (after symmetrizing) by integrable terms due to Lemmas 4.1 and 4.2, respectively. □

5 Bound for the classical region

In this section we shall show that for $\mu > \frac{1}{2}$, $P_1^+ = 0$ with this operator being defined in Lemma 4.4. Notice that it is independent of the particular choice of parameters obeying the conditions of Lemma 4.2 (by the results of Sections 3 and 4). In agreement with Section 2 we may use the notation $P_r^+ = P_1^+$.

Clearly, by symmetry, we may show under the condition (2.3) that $P_l^+ = 0$ if $\mu > \frac{1}{2}$. So we shall here only assume the condition (2.2) (as in the previous sections).

We define

(5.1) $$I_0(t,x) = I(t,x)F_+\big(4\tfrac{x}{t^{\alpha_1}}\big) + I(t,0)F_-\big(4\tfrac{x}{t^{\alpha_1}}\big),$$

(5.2) $$H_0(t) = H + I_0(t,x),$$

and corresponding propagator $U_0(t)$

(5.3) $$i\partial_t U_0(t) = H_0(t)U_0(t),\ U_0(1) = I.$$

Lemma 5.1 *Under the conditions of Lemma 4.2 there exists*

(5.4) $$P^+_{1,0}\phi = \lim_{t\to+\infty} U_0(t)^* F_-\big(|t^\delta H|\big) F_+^2\big(\tfrac{x}{t^{\alpha_1}}\big) F_-^2\big(\tfrac{x}{t^{\alpha_0}}\big) F_-\big(|t^\delta H|\big)\phi(t).$$

Proof Since

$$I_0(t,x)F_+^2\Big(\frac{x}{t^{\alpha_1}}\Big) - F_+^2\Big(\frac{x}{t^{\alpha_1}}\Big)I(t,x) = 0,$$

the proof of Lemma 4.7 carries over. □

A consequence of Lemma 5.1 is that we can assume that P_1^+ is defined by (4.29) with $U(t)$ replaced by $U_0(t)$. In other words, keeping the original notation, we can proceed by showing $P_1^+ = 0$ under the assumption (2.17) for any given $\breve{\alpha} < \alpha_1$ (or $\breve{\alpha} < \alpha$ by adjusting α_1, cf. the proof of Proposition 4.6). Moreover, corresponding to the new $\breve{\alpha}$, we can assume δ is arbitrarily close to $\mu\big(1 + 2^{-1}\breve{\alpha}\big)$, cf. Lemma 2.3. In this section we need the requirements (2.18), (3.1), (3.2) and (4.1)–(4.5) but the latter only for $j = 1$. To be specific we define for sufficiently small $\sigma > 0$ (cf. the proof of Proposition 4.6):

(5.5) $$\begin{aligned} &\alpha_0 = \alpha + \frac{\sigma}{\mu},\\ &\gamma_1 = \frac{\mu}{2}\alpha_0 + \sigma = \gamma + \frac{3}{2}\sigma,\\ &\alpha_1 = 1 - \gamma_1 - \sigma = \alpha - \frac{5}{2}\sigma,\\ &\delta = \frac{1}{2} + \frac{\mu}{2}\alpha,\\ &\beta_1 = \frac{1}{2} + \frac{\mu}{2}\alpha + 2\sigma. \end{aligned}$$

Then all of the above requirements as well as (4.34) and (4.37) are fulfilled.

We shall proceed from here by using the proof of Lemma 4.5 (with $j = 1$) to obtain the representation

(5.6) $$\begin{aligned} &P_1^+ = s - \lim_{t\to+\infty} U(t)^*\Phi(t)U(t),\\ &\Phi(t) = S^* F_-^2\Big(t^{-\beta_1}|t - K(x)|\Big)S, \end{aligned}$$

where $S = S(t)$ is given by (4.36).

We shall use the function K of (4.31). Its inverse is denoted by $L = L_r$. It is characterized as the solution of the initial value problem

(5.7) $$\tfrac{d}{dt}L(t) = \sqrt{-2V(L(t))},\ L(0) = R;\ t \geq 0.$$

Using the upper bound in (4.32) we obtain putting $x = L(t)$ that

(5.8) $$Ct^\alpha \leq L(t);\ t \geq 0.$$

We define the propagator

$$U_r(t) = \exp\left(-i\int_1^t (H + I(s, L(s)))ds\right).$$

We want to prove that there exists

$$(5.9)\qquad P^+_{1,r} \;=\; s-\lim_{t\to+\infty} U_r(t)^*\Phi(t)U(t),$$

where $\Phi(t)$ is specified in (5.6). Since obviously the x–independent potential $I(t,L(t))$ fulfills (2.1) we can use the previously established estimates for the full propagator also for the above auxiliary propagator. Therefore in proving the integrability when differentiating w.r.t. t, it suffices to verify that

$$||(I(t,L(t)) - I(t,x))F^2_-\Big(t^{-\beta_1}|t-K(x)|\Big)||$$

is integrable:

By (2.1), (5.7) and (5.8)

$$\begin{aligned} &I(t,L(t)) - I(t,x) = O\big(t^{-\mu-1}\big)|L(t) - L(K(x))| \\ &\le O\big(t^{-\mu-1}\big)|t-K(x)| \sup_{s\in[t,K(x)]} \sqrt{-2V(L(s))} \\ &\le O\Big(t^{-\mu-1+\beta_1}\Big) \sup_{s\in[t,K(x)]} s^{-\frac{\mu}{2}\alpha} \le Ct^{-\mu-1+\beta_1-\frac{\mu}{2}\alpha}, \end{aligned}$$

where we in the last step used that $\beta_1 < 1$. Since $\mu > \frac{1}{2}$ clearly $\beta_1 < \mu + \frac{\mu}{2}\alpha$, and therefore we have verified the existence of (5.9).

From the existence of $P^+_{1,r}$ it follows readily that indeed $P^+_{1,r} = 0$. As a consequence $P^+_1 = 0$, completing the proof.

In conclusion:

Theorem 5.2 *Under the conditions (2.1) and (2.2) for* $\mu > \frac{1}{2}$

$$P^+_1 = 0.$$

6 Simplification for the inner region

We impose the conditions (2.2) and (2.3). Under the conditions of Lemma 4.2 the statement (2.10) and Lemma 2.3 lead to

$$(6.1)\qquad Q^+_j\phi^+ \;=\; \big(I - P^+_r\big)\phi^+;\; j \;\ge\; 1,$$

with Q^+_j given in Lemma 4.7.

We define

$$(6.2)\qquad I_j(t,x) \;=\; I(t,x)F_-\big(\tfrac{x}{t^{\alpha_{j-1}}}\big) + I(t,0)F_+\big(\tfrac{x}{t^{\alpha_{j-1}}}\big),$$

$$(6.3)\qquad H_j(t) \;=\; H + I_j(t,x),$$

and corresponding propagator $U_j(t)$

$$(6.4)\qquad i\partial_t U_j(t) \;=\; H_j(t)U_j(t),\; U_j(1) \;=\; I.$$

Lemma 6.1 *Under the conditions of Lemma 4.2 there exists for* $j \ge 1$

$$(6.5)\qquad Q^+_{j,j}\phi \;=\; \lim_{t\to+\infty} U_j(t)^*F_-\big(|t^\delta H|\big)F^2_-\big(\tfrac{x}{t^{\alpha_j}}\big)F_-\big(|t^\delta H|\big)\phi(t).$$

Proof We mimic the proof of Lemma 5.1. □

We shall be interested in the sequence $(\alpha_j)_1^\infty$ constructed in the proof of Proposition 4.6 with input $\epsilon > 0$ such that $\alpha_{cr} + \epsilon < \mu$. We fix $j = J$ such that (4.48) holds.

We define

$$I^l(t,x) = I(t,x)F_+\big(-\tfrac{x}{t^{\alpha_J}}\big) + I(t,0)F_-\big(-\tfrac{x}{t^{\alpha_J}}\big), \tag{6.6}$$

$$H^l(t) = H + I^l(t,x), \tag{6.7}$$

and corresponding propagator $U^l(t)$

$$i\partial_t U^l(t) = H^l(t)U^l(t),\ U^l(1) = I. \tag{6.8}$$

Since $\alpha_J < \mu$ and

$$|I^l(t,x) - I_{J+1}(t,x)| \le Ct^{-(\mu+1)+\alpha_J},$$

it follows that there exists the limit

$$s-\lim_{t\to+\infty} U^l(t)^* U_{J+1}(t),$$

cf. the argument leading to (2.17) or (5.9).

Combining this fact, (6.1) and (6.5) for $j = J+1$ we obtain the following result.

Proposition 6.2 *Under the condition (2.2) there exists the limit*

$$\phi^l = \lim_{t\to+\infty} U^l(t)^* U(t)\big(I - P_r^+\big)\phi^+. \tag{6.9}$$

If we also impose the condition (2.3) the previous analysis can be repeated to the left (we may argue by a symmetry argument). But the analogue construction of (6.2) leads to

$$I_j(t,x) = I^l(t,x)F_-\big(-\tfrac{x}{t^{\alpha_{j-1}}}\big) + I^l(t,0)F_+\big(-\tfrac{x}{t^{\alpha_{j-1}}}\big) = I(t,0), \tag{6.10}$$

provided $j \ge J+2$.

Combining Proposition 6.2, (6.10) and an analogue of Lemma 6.1 leads to the existence of (2.9) (for the last conclusion we use [O, Theorem 2.2 p. 196]):

Theorem 6.3 *Under the conditions (2.1)–(2.3) there exist the limits*

$$\begin{aligned}\tilde\phi^+ &= \lim_{t\to+\infty} e^{i\int_1^t (H+I(s,0))ds} U(t)\big(I - P^+\big)\phi^+ \\ &= \lim_{t\to+\infty} e^{i\int_1^t I(s,0)ds} U(t)\big(I - P^+\big)\phi^+\end{aligned} \tag{6.11}$$

(with $P^+ = P_l^+ + P_r^+$). In particular $\tilde\phi^+ = 0$.

In combination with Section 5 we obtain:

Corollary 6.4 *Under the conditions (2.1)–(2.3) with $\mu > \frac{1}{2}$*

$$E_{\{0\}}\big(H^+\big) = 0. \tag{6.12}$$

7 Multi-dimensional spherically symmetric case

In this section we shall consider the n–dimensional case assuming that the potential $V(x)$ is a function of $|x|$ only (and $n \geq 2$). We shall impose the condition (2.1) with derivatives replaced by partial derivatives; for $I(t)$ up to order one and for V to all orders. Again μ is a given fixed number in $(0, \sqrt{3} - 1]$.

We shall impose the following modification of (2.2) and (2.3):

$$V(x) \leq -c|x|^{-\mu}, \ |x| \geq R. \tag{7.1}$$

We define H, $H(t)$, $U(t)$ and the asymptotic energy H^+ as in Section 2 except that the underlying Hilbert space now is $\mathcal{H} = L^2(\mathbf{R}^n_x)$.

We introduce

$$P^+ = s - \lim_{t \to +\infty} U(t)^* E_{[t^{\alpha-\epsilon}, t^{\alpha+\epsilon}]}(|x|) U(t) E_{\{0\}}\big(H^+\big), \tag{7.2}$$

which as in the one-dimensional case can be shown to be well-defined and independent of small $\epsilon > 0$.

The analogues of Theorem 6.3 and Corollary 6.4 are the following results.

Theorem 7.1 *Under the conditions (2.1) (modified) and (7.1) there exist the limits*

$$\begin{aligned} &s - \lim_{t \to +\infty} e^{i\int_0^t (H + I(s,0))ds} U(t)\big(I - P^+\big) E_{\{0\}}\big(H^+\big) \\ &= s - \lim_{t \to +\infty} e^{i\int_0^t I(s,0)ds} U(t)\big(I - P^+\big) E_{\{0\}}\big(H^+\big) = 0. \end{aligned} \tag{7.3}$$

Theorem 7.2 *Under the conditions (2.1) (modified) and (7.1) for $\mu > \frac{1}{2}$*

$$P^+ = 0.$$

We shall embark upon sketching the proofs of the above statements following closely the procedure of the previous sections for the one-dimensional case. At most points only minor modifications are needed.

We write the (minus) Laplacian on $\mathcal{H}$ as

$$p^2 = 4^{-1}\Big(\tfrac{x}{|x|} \cdot p + p \cdot \tfrac{x}{|x|}\Big)^2 + |x|^{-2} L^2 + \tfrac{n-1}{2}\tfrac{n-3}{2}|x|^{-2}, \tag{7.4}$$

where the Laplace–Beltrami operator $L^2 = \sum\limits_{i>j} L^2_{ij}, \ L_{ij} = x_i p_j - x_j p_i$.

Motivated by (7.4) we introduce

$$p_r = \frac{1}{2}(\hat{x} \cdot p + p \cdot \hat{x}), \ \hat{x} = \nabla r, \ r = F_+(|x|)|x|.$$

Due to the spherical symmetry the operator L^2 tends to be "preserved". Hence we may write $H \approx 2^{-1}p_r^2 + V(x)$ in various spherical shells which are analogous to the intervals considered in Section 4. Heuristically, this indicates why the one-dimensional procedure to a large extent works.

We pick a state $\phi^+ \in E_{\{0\}}\big(H^+\big)\mathcal{H}$. The results Lemmas 2.2– 2.4 carry over with almost identical proofs. The same can be said about all results in Section 3. The rule for translating the statements is the following: Replace x by r and p by p_r. Notice for Lemma 3.1 that we use the non-negativity of L^2 and (7.4) to obtain the analogue of (3.9).

To generalize the results of Section 4 we need more refined modifications. We shall assume that $(\lambda_j)_1^\infty$ is a decreasing sequences of positive numbers satisfying

(7.5) $\quad \lambda_j < \alpha_j - \frac{\mu}{2}\alpha_{j-1},$

and

(7.6) $\quad \lambda_j > \alpha_{j-1} - \mu.$

Now, Lemma 4.1 needs to be replaced by the following result.

Lemma 7.3 *Suppose (4.1)–(4.4) and (7.5). Then*

$$(7.7) \quad \begin{aligned} &||\Big\langle t^\delta H\Big\rangle F\Big(\frac{r}{t^{\alpha_j}} > 1, \frac{r}{t^{\alpha_{j-1}}} < 1\Big) F\big(t^{2\gamma_j}p^2 < 1\big) F_-\Big(t^{-2\lambda_j}L^2\Big) F\Big(|t^\delta H| < 1\Big)|| \\ &= O\big(t^{-\infty}\big). \end{aligned}$$

The proof follows that of Lemma 4.1.

We shall need the following two results (with no parallels in Section 4).

Lemma 7.4 *Suppose (7.6), and the following condition on the potential $I(t,x)$ for some $J \in \mathbf{N}$*

(7.8) $\quad I(t,x) = I(t,0)$ *for* $r \geq 2t^{\alpha_{J-1}}.$

Then for any $\phi(t) = U(t)\phi$

(7.9) $\quad \lim\limits_{t\to+\infty} ||F_+\big(t^{-2\lambda_J}L^2\big)\phi(t)|| = 0.$

Proof We use the familiar scheme of the proof of Lemma 3.2 under use of

(7.10) $\quad [I(t,x), L_{ij}] = O\big(t^{-\mu-1+\alpha_{J-1}}\big),$

and (7.6) for $j = J$. □

Lemma 7.5 *Under the conditions of Lemma 7.4, for any $j \leq J$ and $\phi(t) = U(t)\phi$*

(7.11) $\quad \int\limits_1^\infty t^{-1}|\big\langle F\big(t^{-2\lambda_j}L^2 \approx 1\big)\big\rangle_{\phi(t)}|dt \leq C||\phi||^2.$

Proof We mimic the proof of Lemma 2.4 under use of (7.10). □

The analogue of Lemma 4.2 reads:

Lemma 7.6 *Under the conditions (2.18) (with $\breve{\alpha} = 0$), (3.1), (3.2), (4.1)–(4.5), (7.5), (7.6) and (7.8), for any $\phi(t) = U(t)\phi$ and $j \leq J$*

$$H(1,j) \qquad \int\limits_1^\infty t^{-\alpha_j}\big\langle p_r F_+^2(t^{\gamma_j}p_r)\big\rangle_{\tilde{\phi}_j(t)}dt \leq C||\phi||^2,$$

$$H(2,j) \qquad \int\limits_1^\infty t^{-\alpha_j}\big\langle -p_r F_+^2(-t^{\gamma_j}p_r)\big\rangle_{\tilde{\phi}_j(t)}dt \leq C||\phi||^2,$$

where

$$\tilde{\phi}_j(t) = B_j(t)\phi(t);\ B_j(t) = \left(F_+^{2\prime}\right)^{\frac{1}{2}}\Big(\frac{r}{t^{\alpha_j}}\Big)F_-\Big(t^{-2\lambda_j}L^2\Big)F_-\Big(|t^\delta H|\Big).$$

Proof We mimic the proof of Lemma 4.2 proceeding by induction from $j = 1$ to $j = J$. The propagation observable in (4.12) needs to be modified in agreement with the rule mentioned above and in addition be multiplied twice by the factor $F_-\big(t^{-2\lambda_j}L^2\big)$, one from the left and one form the right (for example). The contribution from its Heisenberg derivative is handled by Lemma 7.5, and the substitute for Lemma 4.1 is Lemma 7.3. □

The analogue of Lemma 4.3 reads:

Lemma 7.7 *Under the conditions of Lemma 7.6*

$$\lim_{t\to+\infty} ||F\big(\tfrac{r}{t^{\alpha_j}} > 1, \tfrac{r}{t^{\alpha_{j-1}}} < 1\big)F_-(t^{\gamma_j}p_r)F_-\big(|t^\delta H|\big)\phi(t)|| = 0. \tag{7.12}$$

Proof Using Lemmas 7.4 and 7.3 it suffices to show that

$$\lim_{t\to+\infty} ||F\Big(\frac{r}{t^{\alpha_j}} > 1, \frac{r}{t^{\alpha_{j-1}}} < 1\Big)F_+(-t^{\gamma_j}p_r)F_-\Big(t^{-2\lambda_j}L^2\Big)F_-\Big(|t^\delta H|\Big)\phi(t)|| = 0$$

for which we mimic the proof of Lemma 4.3. □

The next result in Section 4, Lemma 4.4, is modified similarly with obvious proof.

As for Lemma 4.5 the corresponding analogue holds true, but the proof needs a comment: Of course we need to multiply by the factor $F_-\big(t^{-2\lambda_j}L^2\big)$ to get the correct object $B(t,\kappa)$ in (4.33). The proof relies on the factorization $2H \approx p_r^2 + 2V(r) \approx \Big(p_r - (-2V)^{\frac{1}{2}}\Big)\Big(p_r + (-2V)^{\frac{1}{2}}\Big)$, cf. (4.39). Here the last two terms on the right hand side of (7.4) are treated as errors, which is justified if their contributions are integrable. By inspection we need for the latter the condition

$$\beta_j - \mu\alpha_{j-1} + 2(\alpha_j - \lambda_j) > 1 \tag{7.13}$$

in addition to (4.34).

By (7.5) the above conditions are fulfilled for β_j close to one (but smaller). Consequently the rest of the proof of Lemma 4.5 carries over to the present context.

The last result in Section 4, Lemma 4.7, has an obvious analogue with obvious proof.

It remains to prove a statement corresponding to Proposition 4.6. (In the process of doing that we prove Theorem 7.1!) We shall proceed differently verifying inductively the condition (7.8) for a sequence of modified propagators, the latter defined as in Section 6. Precisely we define I_j, H_j and U_j by (6.2), (6.3) and (6.4), respectively (with x replaced by r in (6.2)).

Then by the analogues of Corollary 3.3 and Lemmas 2.4 and 3.4 there exists the limit

$$\phi_1^+ = \lim_{t\to+\infty} U_1(t)^*U(t)\phi^+.$$

(Remember $H^+\phi^+ = 0$.)

For U_1 the statement (7.8) holds for $J = 1$. Consequently we can define the corresponding operator P_1^+. Next we put $\psi_1^+ = \big(I - P_1^+\big)\phi_1^+$. By the definition of P_1^+ we can write $\psi_1^+ = Q_{1,1}^+\phi_1^+$, where $Q_{1,1}^+$ is given as in (6.5). Equivalently we can write

$$\psi_1^+(t) = U_1(t)\psi_1^+ \approx F_-^2\big(\tfrac{r}{t^{\alpha_1}}\big)\psi_1^+(t). \tag{7.14}$$

Obviously we may assume that $(I_2(t,x) - I_1(t,x))F_-^2\left(\frac{r}{t^{\alpha_1}}\right) = 0$ (by using the freedom in defining I_2). Using (7.14) and the modified results of Section 4 discussed above we then get the existence of

$$\psi_2^+ = \lim_{t\to+\infty} U_2(t)^*\psi_1^+(t).$$

Since the statement (7.8) for U_2 holds for $J = 2$ we obtain for the corresponding operator P_2^+ that it is zero (by the analogue of Lemma 4.5). But this implies that

$$\psi_1^+(t) \approx \psi_2^+(t) = U_2(t)\psi_2^+ \approx F_-^2\Big(\frac{r}{t^{\alpha_2}}\Big)\psi_2^+(t).$$

Repeating the procedure yields

$$\psi_1^+(t) \approx \psi_j^+(t) = U_j(t)\psi_j^+ \approx F_-^2\big(\tfrac{r}{t^{\alpha_j}}\big)\psi_j^+(t), \tag{7.15}$$

where by definition

$$\psi_j^+ = \lim_{t\to+\infty} U_j(t)^*\psi_{j-1}^+(t);\ j \geq 2.$$

For $\alpha_j < \mu$ we conclude from (7.15) and by an argument in Section 6 that

$$\psi_1^+(t) \approx \psi_j^+(t) \approx e^{-i\int_1^t (H+I(s,0))ds}\tilde{\psi}^+.$$

By [O, Theorem 2.2 p. 196] it then follows that $\tilde{\psi}^+ = 0$.

It remains to show that indeed we can choose our parameters to obey (2.18) (with $\breve{\alpha} = 0$), (3.1), (3.2), (4.1)–(4.5) (7.5) and (7.6) (and the used condition $\alpha_j < \mu$ for j large enough). For that we mimic the construction in the proof of Proposition 4.6: We keep the first four definitions in (4.44), while the last one is replaced by

$$\alpha_j = \max\Big((\mu+1)^{-1}(\delta+\gamma_j+\sigma), \big(1+\tfrac{\mu}{2}\big)\alpha_{j-1} - \mu + \sigma\Big);\ j \geq 2. \tag{7.16}$$

Then by the proof of Proposition 4.6 the conditions (2.18), (3.1), (3.2) and (4.1)–(4.5) are fulfilled. Notice in particular that $\alpha_2 < \alpha_1$, and therefore that

$$\alpha_j = \max\Big((\mu+1)^{-1}\big(\delta+\tfrac{\mu}{2}\alpha_{j-1}+2\sigma\big), \big(1+\tfrac{\mu}{2}\big)\alpha_{j-1} - \mu + \sigma\Big);\ j \geq 2, \tag{7.17}$$

defines a decreasing sequence with the same limit $\alpha_\infty < \mu$ as before.

Next we need to choose λ_j: By the definition (7.17) (for $j \geq 2$ and (4.44) for $j = 1$)

$$\alpha_j - \tfrac{\mu}{2}\alpha_{j-1} > \alpha_{j-1} - \mu. \tag{7.18}$$

We define J as the largest natural number for which $\alpha_{J-1} - \mu \geq 0$. Then it follows from (7.18) that (7.5) and (7.6) are fulfilled for $j \leq J$ by the construction

$$\lambda_j = \alpha_{j-1} - \mu + \sigma';\ j \leq J,\ \sigma' > 0\ \text{ small.}$$

Notice that the finite sequence it positive and decreasing. We need to supplement by a decreasing sequence of positive λ's with index larger than J. For that we notice that $0 < \alpha_j - \frac{\mu}{2}\alpha_{j-1}$, cf. (4.35). So clearly (7.5) can be fulfilled for a decreasing sequence of positive λ's smaller than λ_J. Since (7.6) is trivial for $j > J$ for the (combined) constructed infinite sequence we finally conclude (7.5) and (7.6).

This completes an outline of the proof of Theorem 7.1.

As for the proof of Theorem 7.2 we mimic Section 5: In addition to (5.5) we define

$$\lambda_1 = \alpha - \frac{1}{2} - 3\sigma.$$

Then (7.5), (7.6) and (7.13) (and previous conditions) are fulfilled for $j = 1$. Moreover with

$$W(t) = \exp\left(-i\int_1^t \left(\frac{1}{2}p_r^2 + V(|x|) + I\left(s, L(s)\frac{x}{|x|}\right)\right)ds\right),$$

where $L(t)$ is given by (5.7), one can define (cf. (5.9))

$$P_{1,W}^{+} = s - \lim_{t\to+\infty} W(t)^{*}\Phi(t)U(t),$$

where $\Phi(t)$ is given by a similar construction as in (5.6) (including two factors of $F_{-}\big(t^{-2\lambda_1}L^2\big)$). Here we use various propagation estimates and that the last two terms on the right hand side of (7.4) are integrable, the latter due to the localization properties of $\Phi(t)$ and the fact that $2(\lambda_1 - \alpha_1) < -1$. Next we notice that $\left(\frac{1}{2}p_r^2 + V(|x|)\right)P_{1,W}^{+} = 0$ yielding to the conclusion that indeed $P_{1,W}^{+} = 0$.

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