# An extension of P. Lévy's distributional properties to the case of a Brownian motion with drift.

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## 1 Introduction

A classical result of Paul Lévy states that if $B = (B_t)_{0 \le t \le 1}$ is a standard *Brownian motion* $(B_0 = 0, EB_t = 0, EB_t^2 = t)$ then

$$(M - B, M) \stackrel{law}{=} (|B|, L(B)) \tag{1}$$

i.e. $((M_t - B_t, M_t); 0 \le t \le 1) \stackrel{law}{=} (|B_t|, L(B)_t; 0 \le t \le 1)$ where $M = (M_t)_{0 \le t \le 1}$, $M_t = \max_{0 \le s \le t} B_s$, and $L(B) = (L(B)_t)_{0 \le t \le 1}$ is the local time of $B$ at zero:

$$L(B)_t = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{(|B_s| \le \epsilon)} \, ds. \tag{2}$$

( See, for example, [1;Ch.VI].)

The main aim of this note is to give an extension of the distributional property (1) to the case of a Brownian motion *with drift* $B^\lambda$ where $B^\lambda = (B_t^\lambda)_{0 \le t \le 1}$, $B_t^\lambda = B_t + \lambda t$. Let's denote $M^\lambda = (M_t^\lambda)_{0 \le t \le 1}$, $M_t^\lambda = \max_{0 \le s \le t} B_s^\lambda$.

For our presentation the following process $X^\lambda = (X_t^\lambda)_{0 \le t \le 1}$ defined as the unique strong solution of the stochastic differential equation

$$dX_t^\lambda = -\lambda \, sgn X_t^\lambda \, dt + dB_t, \quad X_0^\lambda = 0, \tag{3}$$

plays a key role. (Here $sgn\, x$ is defined to be 1 on $\Re_+$, $-1$ on $\Re_-$ and 0 at 0.) In particular we shall see that the process $|X^\lambda| = (|X_t^\lambda|)_{0 \le t \le 1}$ realizes an explicit construction of the process RBM$(-\lambda)$ i.e. a *reflecting Brownian motion with drift* $(-\lambda t)$.

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## 2 Main result

**Theorem 1** *For any* $\lambda \in \Re$

$$(M^\lambda - B^\lambda, M^\lambda) \stackrel{law}{=} (|X^\lambda|, L(X^\lambda)) \tag{4}$$

*i.e.* $(M^\lambda_t - B^\lambda_t, M^\lambda_t); 0 \le t \le 1) \stackrel{law}{=} (|X^\lambda_t|, L(X^\lambda)_t); 0 \le t \le 1)$ *where*

$$L(X^\lambda)_t = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{(|X^\lambda_s| \le \epsilon)} \, ds.$$

Proof. Denote by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le 1}, P)$ a filtered probability space and let $B = (B_t)_{0 \le t \le 1}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le 1}, P)$. Define on $(\Omega, \mathcal{F})$ a new probability measure $P^\lambda$:

$$dP^\lambda = e^{-\lambda B_1 - \lambda^2/2} \, dP \, (= e^{-\lambda B^\lambda_1 + \lambda^2/2} \, dP). \tag{5}$$

By Girsanov's theorem ([1],[3])

$$\mathrm{Law}(B^\lambda \,|\, P^\lambda) = \mathrm{Law}(B \,|\, P). \tag{6}$$

Denoting by $C^+[0,1]$ the space of non-negative continuous functions on $[0,1]$ we obtain, using (5), (6) and (1) that for any non-negative measurable functional $G = G(x,y), \ (x,y) \in C^+[0,1] \times C^+[0,1]$:

$$E[\,G(M^\lambda - B^\lambda, M^\lambda)] = E^\lambda[\,e^{\lambda B^\lambda_1 - \lambda^2/2} G(M^\lambda - B^\lambda, M^\lambda)]$$

$$= E[\,e^{\lambda B_1 - \lambda^2/2} G(M - B, M)] = E[\,e^{\lambda(L(B)_1 - |B_1|) - \lambda^2/2} G(|B|, \, L(B))]. \tag{7}$$

¿From another side let us introduce a new measure $\tilde{P}^\lambda$:

$$d\tilde{P}^\lambda = e^{\lambda \int_0^1 sgn X^\lambda_s \, dB_s - \lambda^2/2} \, dP \, (= e^{\lambda \int_0^1 sgn X^\lambda_s \, dX^\lambda_s + \lambda^2/2} \, dP) \tag{8}$$

Again by Girsanov's theorem

$$\mathrm{Law}(X^\lambda \,|\, \tilde{P}^\lambda) = \mathrm{Law}(B \,|\, P). \tag{9}$$

¿From (8) and (9) we find that (with $\tilde{E}^\lambda$ denoting expectation w.r. to $\tilde{P}^\lambda$)

$$E[\,G(|X^\lambda|, L(X^\lambda))] = \tilde{E}^\lambda[e^{-\lambda \int_0^1 sgn X^\lambda_s \, dX^\lambda_s - \lambda^2/2} \, G(|X^\lambda|, L(X^\lambda))]$$

$$= E[\,e^{-\lambda \int_0^1 sgn B_s \, dB_s - \lambda^2/2} \, G(|B|, L)]. \tag{10}$$

Now we note that by Tanaka's formula [1; Ch. VI]

$$|B_t| = \int_0^1 sgn B_s \, dB_s + L(B)_t.$$

So, from (10)

$$E[G(|X^\lambda|, L(X^\lambda))] = E[e^{\lambda(L(B)_1 - |B_1|) - \lambda^2/2} \, G(|B|, L(B))]. \tag{11}$$

Comparing (7) and (11) we get the statement (4).

## 3 Study of $X^\lambda$

In this section we consider some properties of the processes $X^\lambda$ and $|X^\lambda|$. If $\lambda = 0$ then $X^0 = B$, $|X^0| = |B|$ and as it is well-known Law($|B|$)=Law(RBM(0)), where RBM(0) is a *Brownian motion reflecting in zero*; [1;Ch.III], [2;Ch.IV]. In this sense the process $|B|$ gives an *explicit* construction of the reflecting Brownian motion. We shall see below that for reflecting Brownian with drift the process $|X^\lambda|$ plays the corresponding role.

Let's describe first of all some properties of $X^\lambda$ and $|X^\lambda|$ from the point of view of the general theory of Markov processes.

On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ for given $\lambda \in \Re$ and every $x \in \Re$ we consider the stochastic process $X^{x,\lambda} = (X_t^{x,\lambda})_{t\geq 0}$ which satisfies the stochastic differential equation

$$dX_t^{x,\lambda} = -\lambda sgn X_t^{x,\lambda}\, dt + dB, \quad X_0^{x,\lambda} = x. \tag{12}$$

This equation has a unique strong solution and as a corollary (see [1;Ch.IX, Th. 1.11]) we also have uniqueness in law. Denote the corresponding distribution of $X^{x,\lambda}$ on the space $(C, \mathcal{C})$ of continuous functions by $P^{x,\lambda}$:

$$\text{Law}(X^{x,\lambda} |\, P) = P^{x,\lambda}. \tag{13}$$

Denote also by $(T_t^\lambda,\, t \geq 0)$ the set of operators given by

$$T_t^\lambda f(x) = \int f(c_t)\, P^{x,\lambda}(dc), \tag{14}$$

where $f \in \mathcal{B}_b(\Re)$ (the set of bounded Borel measurable real valued functions defined on $\Re$) and $c = (c_t)_{t\geq 0}$ denotes the coordinate process, $c \in C$.

If $\tau$ is a finite $(\mathcal{F}_t)_{t\geq 0}$-stopping time and $A \in \mathcal{F}_\tau$ then

$$E[\, f(X_{\tau+t}^{x,\lambda}) \cdot \mathbf{1}_A] = E[\, T_t f(X_\tau^{x,\lambda}) \cdot \mathbf{1}_A]. \tag{15}$$

Indeed, from (12)

$$X_{\tau+t}^{x,\lambda} = X_\tau^{x,\lambda} - \lambda \int_0^t sgn(X_{\tau+u}^{x,\lambda})\, du + (B_{\tau+t} - B_\tau). \tag{16}$$

But Law$(B_{\tau+t} - B_\tau,\ t \geq 0 |\, P)$ =Law$(B_t,\ t \geq 0 |\, P)$ and $(B_{\tau+t} - B_\tau)_{t\geq 0}$ is independent of $\mathcal{F}_\tau$ and so by uniqueness in law of equation (12) we get (15).

Thus the process $X^{x,\lambda} = (X_t^{x,\lambda})_{t\geq 0}$ is a time homogeneous *Markov process* with transition functions $(T_t^\lambda,\, t \geq 0)$. ¿From [4; Ch.6,6.5] it follows that transition densities $p_t^\lambda(y|x)$ such that

$$T_t^\lambda(x, dy) = p_t^\lambda(y|x)\, dy$$

do exist and, for example, for $x \geq 0,\ \lambda \geq 0$ are given by the following formula:

$$p_t^\lambda(y|x) = \frac{1}{\sqrt{2\pi t}}\left(e^{-\frac{(x-y-\lambda t)^2}{2t}} + \lambda e^{-2\lambda y}\int_{x+y}^{\infty} e^{-\frac{(v-\lambda t)^2}{2t}}\,dv\right), \quad y \geq 0,$$

$$p_t^\lambda(y|x) = \frac{1}{\sqrt{2\pi t}}\left(e^{-(2\lambda x-\frac{(x-y+\lambda t)^2}{2t})} + \lambda e^{2\lambda y}\int_{x-y}^{\infty} e^{-\frac{(v-\lambda t)^2}{2t}}\,dv\right), \quad y < 0. \tag{17}$$

This explicit form of the transition density can be used to show that $X^{x,\lambda}$ is a *Feller process* (indeed that can also be deduced using Zvonkin's method [1;Ch.IX, (2.11)].)

Now we show that $|X^{x,\lambda}|$ is also a time homogeneous Markov process.

Indeed $sgn\, x$ is an odd function and $\{t\,|\,X_t^{x,\lambda}=0\}$ is $P$-a.s. a Lebesgue null set (it is clearly true for $\lambda = 0$, that is for $(x+B_t)_{t\geq 0}$, but the measures $P^{x,0}$ and $P^{x,\lambda}$ are locally equivalent so it holds, in fact, for any $\lambda \in \Re$). Thus it follows that $P$-a.s.

$$-X_t^{x,\lambda} = -x - \lambda\int_0^t sgn(-X_s^{x,\lambda})\,ds - B_t, \tag{18}$$

and by the uniqueness in law we then obtain

$$\mathrm{Law}(-X^{x,\lambda}\,|\,P) = \mathrm{Law}(X^{-x,\lambda}\,|\,P). \tag{19}$$

Using the Markov property of $X^{x,\lambda}$-processes this implies that for all $s,\ t \geq 0,\ x \in [0,\infty)$ and all bounded real valued Borel function $f$ on $[0,\infty)$ we have for any $A^x \in \sigma(|X_u^{x,\lambda}|\,|\,u \leq s\,)$

$$E[\,f(|X_{s+t}^{x,\lambda}|), A^x] = E[\,\tilde f(X_{s+t}^{x,\lambda}), A^x] = E[\,T_t\tilde f(X_s^{x,\lambda}), A^x]$$

and

$$E[\,f(|X_{s+t}^{x,\lambda}|), A^x] = E[\,f(|-X_{s+t}^{x,\lambda}|), A^x] = E[\,f(|X_{s+t}^{-x,\lambda}|), A^{-x}]$$

$$= E[\,\tilde f(X_{s+t}^{-x,\lambda}), A^{-x}] = E[\,T_t\tilde f(X_s^{-x,\lambda}), A^{-x}] = E[\,T_t\tilde f(-X_s^{x,\lambda}), A^x].$$

Here we have used the notation $\tilde f(x)$ for $f(|x|),\ x \in \Re$. In other words

$$E[\,f(|X_{s+t}^{x,\lambda}|), A] = E[\,T_t\tilde f(|X_s^{x,\lambda}|), A] \tag{20}$$

showing that $|X^{x,\lambda}|$ is indeed a Feller Markov process.

**Theorem 2** *For each $x \in \Re_+$ and $\lambda \in \Re$*

$$\mathrm{Law}(|X^{x,\lambda}|) = \mathrm{Law}(\mathrm{RBM}^x(-\lambda)) \tag{21}$$

Proof. In Markov theory, as it is well-known (see, for example, [2;Ch.IV,§5]), the process $\mathrm{RBM}^x(-\lambda)$, called a *Brownian motion with drift* $(-\lambda t)$ *started at* $x \geq 0$ *and reflected at zero*, is a diffusion Markov process with infinitesimal operator $\mathcal{A}^\lambda$ acting on functions

$$\mathcal{D}(\mathcal{A}^\lambda) = \{f \in C_b^2\left([0,\infty)\right), \frac{df}{dx}|_{x\downarrow 0} = 0\}$$

by the formula

$$\mathcal{A}^\lambda f(x) = \frac{1}{2} f''(x) - \lambda f'(x). \tag{22}$$

(It is well-known that the operator $\mathcal{A}^\lambda$ generates a unique (diffusion) family of measures $Q^{x,\lambda}$, $x \geq 0$ and the corresponding Markov process, is by definition the process $\mathrm{RBM}^x(-\lambda)$, [2].)

Now let's consider our process $X^{x,\lambda}$. By the Itô-Tanaka formula, [1;Ch.VI]

$$\begin{aligned} d|X_t^{x,\lambda}| &= sgn X_t^{x,\lambda}\, dX_t^{x,\lambda} + dL(X^{x,\lambda})_t \\ &= -\lambda\, dt + sgn X_t^{x,\lambda}\, dB_t + dL(X^{x,\lambda})_t \end{aligned} \tag{23}$$

where $L(X^{x,\lambda})_t$ is a local time at zero on the time interval $[0,t]$ for the process $X^{x,\lambda}$. Suppose that $f \in C_b^2([0,\infty))$ with $f'(0+) = \frac{df}{dx}|_{x\downarrow 0} = 0$. Then by Itô's formula

$$f(|X_t^{x,\lambda}|) - f(|X_0^{x,\lambda}) = \int_0^t f'(|X_s^{x,\lambda}|)\, d|X_s^{x,\lambda}| + 1/2 \int_0^t f''(|X_s^{x,\lambda}|)\, ds$$

$$= \int_0^t f'(|X_s^{x,\lambda}|)\, (-\lambda\, ds + sgn X_s^{x,\lambda}\, dB_s + dL(X^{x,\lambda})_s) + 1/2 \int_0^t f''(|X_s^{x,\lambda}|)\, ds \tag{24}$$

$$= \int_0^t (-\lambda f'(X_s^{x,\lambda}|) f''(|X_s^{x,\lambda}|)\, ds + M_t$$

where $M_t = \int_0^t f'(|X_s^{x,\lambda}|) sgn X_s^{x,\lambda}\, dB_s$ is a local martingale. Note that

$$\int_0^t f'(|X_s^{x,\lambda}|)\, dL(X^{x,\lambda})_s = 0$$

because $f'(0+) = 0$ and $L(X^{x,\lambda})$ increases only on the time set $\{t\,|\, X_t^{x,\lambda} = 0\}$. From (24) we see that

$$f(|X_t^{x,\lambda}|) - f(|X_0^{x,\lambda}|) - \int_0^t \mathcal{A}^\lambda f(|X_s^{x,\lambda}|)\, ds \tag{25}$$

is a local martingale and thus the infinitesimal operators for the two processes $|X^{x,\lambda}|$ and $\mathrm{RBM}^x(-\lambda)$ are the same (acting on $\mathcal{D}(\mathcal{A}^\lambda)$). Therefore (21) is proved.

# References


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