SEMIGROUPS AND PROCESSES WITH PARAMETER IN A CONE

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Abstract. For a cone K in a Euclidean space recent results of Pedersen and Sato on K-parameter convolution semigroups and K-parameter Lévy processes in law are surveyed. Relations to other multi-parameter processes are discussed.

1. Convolution semigroups with parameter in a cone

A convolution semigroup with parameter in $\mathbb{R}_+ = [0, \infty)$, or an $\mathbb{R}_+$-parameter convolution semigroup, is a family of probability measures $\{\mu_t \colon t \in \mathbb{R}_+\}$ on $\mathbb{R}^d$ such that

$$\mu_{t_1} * \mu_{t_2} = \mu_{t_1+t_2} \quad \text{for } t_1, t_2 \in \mathbb{R}_+ \tag{1.1}$$

and

$$\mu_{t_n} \to \delta_0 \quad \text{whenever } t_n \downarrow 0. \tag{1.2}$$

Here $\mu_{t_1} * \mu_{t_2}$ is the convolution of μ_{t_1} and μ_{t_2} defined by

$$(\mu_{t_1} * \mu_{t_2})(B) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} 1_B(x+y)\mu_{t_1}(dx)\mu_{t_2}(dy),$$

where 1_B is the indicator function of the set B; the convergence $\mu_{t_n} \to \delta_0$ is weak convergence and δ_0 is the distribution concentrated at 0, that is,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f(x)\mu_{t_n}(dx) = f(0) \quad \text{for all bounded continuous } f(x).$$

The structure of $\mathbb{R}_+$-parameter convolution semigroups is as follows. We use the notion of infinite divisibility. A distribution (= probability measure) μ on $\mathbb{R}^d$ is called infinitely divisible if, for every positive integer n, there is a distribution ρ_n such that $\mu = {\rho_n}^{n*}$, the n-fold convolution of ρ_n.

This work was partially supported by MaPhySto – A Network in Mathematical Physics and Stochastics, funded by The Danish National Research Foundation. Jan Pedersen was supported by the Danish Natural Science Research Council. Ken-iti Sato was partly supported by CIMAT at Guanajuato, Mexico, and the Grant-in-Aid for Scientific Research, Japan.

(i) If $\{\mu_t \colon t \in \mathbb{R}_+\}$ is an $\mathbb{R}_+$-parameter convolution semigroup on $\mathbb{R}^d$, then μ_1 is obviously infinitely divisible; conversely, if μ is infinitely divisible, then there is a unique $\mathbb{R}_+$-parameter convolution semigroup $\{\mu_t \colon t \in \mathbb{R}_+\}$ such that $\mu_1 = \mu$.

(ii) If μ is an infinitely divisible distribution on $\mathbb{R}^d$, then the characteristic function $\widehat{\mu}(z) = \int_{\mathbb{R}^d} e^{i\langle z,x\rangle}\mu(dx)$, $z \in \mathbb{R}^d$, of μ has the representation

$$\widehat{\mu}(z) = \exp\left[-\frac{1}{2}\langle z, Az\rangle + \int_{\mathbb{R}^d} g(z,x)\nu(dx) + i\langle z,\gamma\rangle\right] \tag{1.3}$$

with

$$g(z,x) = e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle 1_{\{|x|\leqslant 1\}}(x),$$

where $1_{\{|x|\leqslant 1\}}(x)$ is the indicator function of the set $\{|x| \leqslant 1\}$. Here $A \in \mathbf{S}_d^+ = \{$ nonnegative-definite symmetric $d \times d$ matrices $\}$, ν is a measure on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int (1 \wedge |x|^2)\nu(dx) < \infty$, and $\gamma \in \mathbb{R}^d$. These A, ν, and γ are uniquely determined by μ and called the (generating) triplet of μ; A is the Gaussian covariance matrix, ν is the Lévy measure, and γ is a location parameter of μ. Conversely, for any A, ν, and γ satisfying the conditions above, there is a unique infinitely divisible distribution μ on $\mathbb{R}^d$ having the representation (1.3). (This is called the Lévy-Khintchine representation of μ).

In fact, if $\{\mu_t \colon t \in \mathbb{R}_+\}$ is an $\mathbb{R}_+$-parameter convolution semigroup on $\mathbb{R}^d$, then, for each t, μ_t is infinitely divisible and has triplet $(tA, t\nu, t\gamma)$, where (A,ν,γ) is the triplet of μ_1.

These results, and their probabilistic interpretations, were obtained in the 1930s by de Finetti, Kolmogorov, Lévy, and Khintchine; see [3], [4], [23], [24].

A natural generalization of the parameter space $\mathbb{R}_+$ is $\mathbb{R}_+^N$, the set of $s = (s_1,\ldots,s_N)^\top$ with $s_1,\ldots,s_N \in \mathbb{R}_+$. The notation $(s_1,\ldots,s_N)^\top$ gives the column vector with components $s_1,\ldots,s_N$. We say that a family $\{\mu_s \colon s \in \mathbb{R}_+^N\}$ of probability measures on $\mathbb{R}^d$ is an $\mathbb{R}_+^N$-parameter convolution semigroup if

$$\mu_{s^1} * \mu_{s^2} = \mu_{s^1+s^2} \quad \text{for } s^1, s^2 \in \mathbb{R}_+^N \tag{1.4}$$

and

$$\mu_{t_n s} \to \delta_0 \quad \text{for } s \in \mathbb{R}_+^N \text{ whenever } \{t_n\} \text{ is a sequence of reals strictly decreasing to } 0. \tag{1.5}$$

Although $\mathbb{R}_+^N$-parameter semigroups of linear operators on a Banach space are discussed in Dunford and Schwartz [6], we do not know any reference for the explicit statement of the following result.

Theorem 1.1. *Let $e^j = (\delta_{jk})_{1\leqslant k\leqslant N}$, where $\delta_{jk} = 0$ or 1 according as $k \neq j$ or $k = j$.*

(i) *Let $\{\mu_s \colon s \in \mathbb{R}_+^N\}$ be an $\mathbb{R}_+^N$-parameter convolution semigroup on $\mathbb{R}^d$. Then, for any $s = s_1 e^1 + \cdots + s_N e^N \in \mathbb{R}_+^N$, μ_s is infinitely divisible and the triplet (A_s, ν_s, γ_s) of μ_s is such that*

$$A_s = s_1 A_{e^1} + \cdots + s_N A_{e^N}, \quad \nu_s = s_1 \nu_{e^1} + \cdots + s_N \nu_{e^N}, \quad \gamma_s = s_1 \gamma_{e^1} + \cdots + s_N \gamma_{e^N}.$$

(ii) *Conversely, let $\rho_1, \dots, \rho_N$ be infinitely divisible distributions on $\mathbb{R}^d$ with triplets $(A_1, \nu_1, \gamma_1), \dots, (A_N, \nu_N, \gamma_N)$, respectively. Then there is a unique $\mathbb{R}_+^N$-parameter convolution semigroup $\{\mu_s \colon s \in \mathbb{R}_+^N\}$ such that $\mu_{e^j} = \rho_j$ for $j = 1, \dots, N$.*

Proof. (i) Notice that the restriction of $\{\mu_s\}$ to $s = te^j$, $t \in \mathbb{R}_+$, is an $\mathbb{R}_+$-parameter convolution semigroup for each j and that $\mu_s = \mu_{s_1 e^1} * \cdots * \mu_{s_N e^N}$ for $s = s_1 e^1 + \cdots + s_N e^N \in \mathbb{R}_+^N$. (ii) Let $\{\mu_t^{(j)} \colon t \in \mathbb{R}_+\}$ be the $\mathbb{R}_+$-parameter convolution semigroup with $\mu_1^{(j)} = \rho_j$ and consider $\mu_s = \mu_{s_1}^{(1)} * \cdots * \mu_{s_N}^{(N)}$ for $s = s_1 e^1 + \cdots + s_N e^N \in \mathbb{R}_+^N$. □

A further natural generalization of the parameter space is a cone K in a Euclidean space $\mathbb{R}^M$. We say that a subset K of $\mathbb{R}^M$ is a *cone* if it is a non-empty closed convex set which is closed under multiplication by nonnegative reals and contains no straight line through 0 and if $K \neq \{0\}$. Given a cone K, we call $\{\mu_s \colon s \in K\}$ a *K-parameter convolution semigroup* if it is a family of probability measures on $\mathbb{R}^d$ satisfying (1.4) and (1.5) with $\mathbb{R}_+^N$ replaced by K. Bochner introduced in his book [4] a similar concept and suggested subordination of cone-parameter convolution semigroups, in a short section entitled *Multidimensional time variable.*

Let K be a cone in $\mathbb{R}^M$ and let L be the linear subspace generated by K. If L has dimension N, then K is called an N-dimensional cone. If L has dimension N and $\{e^1, \dots, e^N\}$ is a linearly independent system such that each e^j is in K, then we call $\{e^1, \dots, e^N\}$ a *weak basis* of K. If, moreover, each $s \in K$ is represented as $s = s_1 e^1 + \cdots + s_N e^N$ with $s_1, \dots, s_N \in \mathbb{R}_+$, then we call $\{e^1, \dots, e^N\}$ a *strong basis* of K. If K is a cone with a strong basis $\{e^1, \dots, e^N\}$, then K is isomorphic to $\mathbb{R}_+^N$. Any 2-dimensional cone has a strong basis, but in higher dimensions there are many cones without a strong basis. Indeed, a 3-dimensional cone has a strong basis if and only if it is a triangular cone.

The latter half of Theorem 1.1 is not generalized for K-parameter convolution semigroups unless K has a strong basis. For an infinitely divisible distribution μ on

$\mathbb{R}^d$ satisfying (1.3) we define

$$\widehat{\mu}(z)^t = \exp\left[t\left(-\frac{1}{2}\langle z, Az\rangle + \int_{\mathbb{R}^d} g(z,x)\nu(dx) + i\langle z,\gamma\rangle\right)\right]$$

for $t \in \mathbb{R}$. If $A \neq 0$ or $\nu \neq 0$, then, for any $t < 0$, $\widehat{\mu}(z)^t$ is not a characteristic function. Let $\mathcal{B}(\mathbb{R}^d)$ be the class of Borel sets in $\mathbb{R}^d$ and let $\mathcal{B}_0(\mathbb{R}^d)$ be the class of $B \in \mathcal{B}(\mathbb{R}^d)$ such that $\inf_{x\in B}|x| > 0$.

Theorem 1.2. *Let K be a cone and let $\{e^1,\dots,e^N\}$ be a weak basis.*

(i) *Let $\{\mu_s\colon s \in K\}$ be a K-parameter convolution semigroup on $\mathbb{R}^d$. Then, for any $s \in K$, μ_s is infinitely divisible and determined by $\mu_{e^1},\dots,\mu_{e^N}$. More precisely, for $s = s_1e^1 + \cdots + s_Ne^N \in K$, we have*

$$\widehat{\mu}_s(z) = \widehat{\mu}_{e^1}(z)^{s_1}\cdots\widehat{\mu}_{e^N}(z)^{s_N}, \quad z \in \mathbb{R}^d \tag{1.6}$$

and the triplet (A_s,ν_s,γ_s) of μ_s satisfies

$$A_s = s_1A_{e^1} + \cdots + s_NA_{e^N}, \tag{1.7}$$

$$\nu_s(B) = s_1\nu_{e^1}(B) + \cdots + s_N\nu_{e^N}(B) \quad \text{for } B \in \mathcal{B}_0(\mathbb{R}^d), \tag{1.8}$$

$$\gamma_s = s_1\gamma_{e^1} + \cdots + s_N\gamma_{e^N}. \tag{1.9}$$

(ii) *Let $\rho_1,\dots,\rho_N$ be infinitely divisible distributions on $\mathbb{R}^d$. Then the following conditions are equivalent.*

(a) *There exists a K-parameter convolution semigroup $\{\mu_s\colon s \in K\}$ such that $\mu_{e^j} = \rho_j$ for $j = 1,\dots,N$.*

(b) *If $s_1,\dots,s_N \in \mathbb{R}$ are such that $s_1e^1+\cdots+s_Ne^N \in K$, then $\widehat{\rho}_1(z)^{s_1}\dots\widehat{\rho}_N(z)^{s_N}$ is an infinitely divisible characteristic function.*

(c) *If $s_1,\dots,s_N \in \mathbb{R}$ are such that $s_1e^1+\cdots+s_Ne^N \in K$, then $s_1A_1+\cdots+s_NA_N \in \mathbf{S}_d^+$ and $s_1\nu_1(B) + \cdots + s_N\nu_N(B) \geqslant 0$ for $B \in \mathcal{B}_0(\mathbb{R}^d)$.*

The proof of this theorem is not hard. We refer to [20]. Note that some of $s_1,\dots,s_N$ above may be negative.

A system $\{\rho_1,\dots,\rho_N\}$ of infinitely divisible distributions on $\mathbb{R}^d$ is called *admissible* with respect to a weak basis $\{e^1,\dots,e^N\}$, if it satisfies condition (a) above. If $\{e^1,\dots,e^N\}$ is a strong basis of K, then every system $\{\rho_1,\dots,\rho_N\}$ is admissible with respect to $\{e^1,\dots,e^N\}$ by virtue of Theorem 1.1. The converse is also true, as we formulate it below.

Corollary 1.3. *Let $\{e^1, \ldots, e^N\}$ be a weak basis of K and not a strong basis. Then, for any d, there is a system $\{\rho_1, \ldots, \rho_N\}$ of infinitely divisible distributions on $\mathbb{R}^d$ which is not admissible with respect to $\{e^1, \ldots, e^N\}$.*

Proof. Let the Lévy measures ν_j of ρ_j $(j = 1, \ldots, N)$ be such that there is $B_j \in \mathcal{B}_0(\mathbb{R}^d)$ satisfying $\nu_j(B_j) > 0$ and $\nu_k(B_j) = 0$ for $k \neq j$. Since $\{e^1, \ldots, e^N\}$ is not a strong basis, we can find $s = s_1 e^1 + \cdots + s_N e^N \in K$ such that $s_{j_0} < 0$ for some j_0. Then

$$s_1\nu_1(B_{j_0}) + \cdots + s_N\nu_N(B_{j_0}) = s_{j_0}\nu_{j_0}(B_{j_0}) < 0,$$

which shows that $\{\rho_1, \ldots, \rho_N\}$ is not admissible. □

We have the following result concerning the continuity condition in the definition of cone-parameter convolution semigroups.

Corollary 1.4. *Let K be a cone. Let $\{\mu_s \colon s \in K\}$ be a K-parameter convolution semigroup on $\mathbb{R}^d$. Then, for any $s \in K$,*

$$\mu_{s'} \to \mu_s \quad \text{whenever } s' \to s \text{ in } K. \tag{1.10}$$

Proof. Use (1.6). □

Let K_1 and K_2 be cones in $\mathbb{R}^M$ satisfying $K_1 \subseteq K_2$. It is obvious that if $\{\mu_s \colon s \in K_2\}$ is a K_2-parameter convolution semigroup, then its restriction $\{\mu_s \colon s \in K_1\}$ is a K_1-parameter convolution semigroup. Concerning the extension problem we have the following.

Theorem 1.5. *Let K_1 be an N-dimensional cone with a strong basis. Then, for any d, there exists a K_1-parameter convolution semigroup $\{\mu_s \colon s \in K_1\}$ on $\mathbb{R}^d$ such that, for any N-dimensional cone K_2 satisfying $K_2 \supseteq K_1$ and $K_2 \neq K_1$, $\{\mu_s \colon s \in K_1\}$ is not extendable to a K_2-parameter convolution semigroup.*

Proof. Construct $\{\mu_s \colon s \in K_1\}$ similarly to the proof of Corollary 1.3. □

Example 1.6. The class $\mathbf{S}_d^+$ is a $(d(d+1)/2)$-dimensional cone. For any $d \geqslant 2$ it has no strong basis. The class $\mathbf{S}_2^+$ is 3-dimensional and isomorphic to a circular cone. See [20] for a proof.

Let $\mu_s = N_d(0, s)$, the Gaussian distribution on $\mathbb{R}^d$ with mean 0 and covariance matrix s. Then $\{\mu_s \colon s \in \mathbf{S}_d^+\}$ is an $\mathbf{S}_d^+$-parameter convolution semigroup. This is introduced in [20] and called the *canonical* $\mathbf{S}_d^+$*-parameter convolution semigroup.*

Let K_1 and K_2 be cones in $\mathbb{R}^{M_1}$ and $\mathbb{R}^{M_2}$, respectively. A K_1-parameter convolution semigroup $\{\mu_s \colon s \in K_1\}$ on $\mathbb{R}^{M_2}$ is called *K_2-valued* if, for each s, μ_s has support in K_2. The condition for an infinitely divisible distribution to have support in a cone is given in Skorohod [26] (see [23], E 22.11). Many examples of $\mathbb{R}_+^N$-valued $\mathbb{R}_+$-parameter convolution semigroups are given in the paper [2]. Now let us consider an extension of Bochner's subordination to the cone-parameter case. For any measure μ and μ-integrable function f, we write $\mu(f) = \int f(x)\mu(dx)$.

Theorem 1.7. *Let $\{\mu_u \colon u \in K_2\}$ be a K_2-parameter convolution semigroup on $\mathbb{R}^d$ and let $\{\rho_s \colon s \in K_1\}$ be a K_2-valued K_1-parameter convolution semigroup. Define a probability measure σ_s on $\mathbb{R}^d$ by*

$$\sigma_s(f) = \int_{K_2} \mu_u(f)\rho_s(du), \qquad s \in K_1 \tag{1.11}$$

for bounded continuous functions f on $\mathbb{R}^d$. Then $\{\sigma_s \colon s \in K_1\}$ is a K_1-parameter convolution semigroup on $\mathbb{R}^d$.

Proof. We have

$$\widehat{\sigma}_s(z) = \int_{K_2} \widehat{\mu}_u(z)\rho_s(du), \qquad z \in \mathbb{R}^d.$$

Hence $\lim_{n\to\infty} \widehat{\sigma}_{t_n s}(z) = 1$ as $t_n \downarrow 0$ and

$$\begin{aligned}
\widehat{\sigma}_{s^1+s^2}(z) &= \int_{K_2} \widehat{\mu}_u(z)\rho_{s^1+s^2}(du) = \iint_{K_2\times K_2} \widehat{\mu}_{u^1+u^2}(z)\rho_{s^1}(du^1)\rho_{s^2}(du^2) \\
&= \iint_{K_2\times K_2} \widehat{\mu}_{u^1}(z)\widehat{\mu}_{u^2}(z)\rho_{s^1}(du^1)\rho_{s^2}(du^2) = \widehat{\sigma}_{s^1}(z)\widehat{\sigma}_{s^2}(z).
\end{aligned}$$

Thus $\{\sigma_s\}$ is a K_1-parameter convolution semigroup. □

The expression of the triplet of σ_s of Theorem 1.7 in terms of the triplets of μ_u, $u \in K_2$, and ρ_s is given in [20].

2. Lévy processes in law with parameter in a cone

To any $\mathbb{R}_+$-parameter convolution semigroup a unique (in law) Lévy process in law is associated and vice versa. More precisely, a stochastic process $\{X_t \colon t \in \mathbb{R}_+\}$

on $\mathbb{R}^d$ is called a Lévy process in law if

(2.1) for any $n \geqslant 3$ and $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$ in $\mathbb{R}_+$, $X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent,

(2.2) $X_{t_2} - X_{t_1} \stackrel{\mathrm{d}}{=} X_{t_4} - X_{t_3}$ for $t_2 - t_1 = t_4 - t_3$,

(2.3) $X_0 = 0$ a. s.(= almost surely),

(2.4) $X_{t_n} \to X_t$ in probability as $|t_n - t| \to 0$.

Here (and from now on) we do not explicitly mention the probability space $(\Omega, \mathcal{F}, P)$ on which the process is defined. The symbol $\stackrel{\mathrm{d}}{=}$ indicates equality in distribution. Let us denote the distribution of a random vector X by $\mathcal{L}(X)$. If $\{X_t \colon t \in \mathbb{R}_+\}$ is a Lévy process in law on $\mathbb{R}^d$, then $\{\mu_t \colon t \in \mathbb{R}_+\}$ defined by $\mu_t = \mathcal{L}(X_t)$ is an $\mathbb{R}_+$-parameter convolution semigroup on $\mathbb{R}^d$. Conversely, for any $\mathbb{R}_+$-parameter convolution semigroup $\{\mu_t \colon t \in \mathbb{R}_+\}$, there exists a unique (in law) Lévy process in law $\{X_t \colon t \in \mathbb{R}_+\}$ such that $\mathcal{L}(X_t) = \mu_t$.

We pose the following problem. For a cone K in $\mathbb{R}^M$, can we define a class of K-parameter stochastic processes which corresponds to the class of K-parameter convolution semigroups? When a K-parameter convolution semigroup $\{\mu_s \colon s \in K\}$ on $\mathbb{R}^d$ is given, we mean

(2.5) $$\mathcal{L}(X_s) = \mu_s \quad \text{for } s \in K$$

by saying that a process $\{X_s \colon s \in K\}$ corresponds to $\{\mu_s \colon s \in K\}$. Since $\mu_0 = \delta_0$, (2.5) implies

(2.6) $$X_0 = 0 \quad \text{a. s.}$$

We have to seek probabilistic properties of $\{X_s \colon s \in K\}$ which gives the semigroup property

(2.7) $$\mu_{s^1} * \mu_{s^2} = \mu_{s^1+s^2} \quad \text{for } s^1, s^2 \in K.$$

As is shown in Examples 2.15–2.16 below, (2.7) is not satisfied for many $\mathbb{R}_+^N$-parameter processes studied in the literature if μ_s is given by (2.5).

A cone K in $\mathbb{R}^M$ induces a partial order in $\mathbb{R}^M$; we say that $s^1 \leqslant_K s^2$ if $s^2 - s^1 \in K$. The following two conditions combined with (2.6) imply the semigroup property (2.7) if μ_s is given by (2.5):

(2.8) if $n \geqslant 3$ and $s^1 \leqslant_K s^2 \leqslant_K \cdots \leqslant_K s^n$, then $X_{s^2} - X_{s^1}, \ldots, X_{s^n} - X_{s^{n-1}}$ are independent,

(2.9) if $s^1, \ldots, s^4 \in K$ and $s^2 - s^1 = s^4 - s^3 \in K$, then $X_{s^2} - X_{s^1} \stackrel{\mathrm{d}}{=} X_{s^4} - X_{s^3}$.

The following condition implies the continuity property (1.10):

(2.10) $X_{s^n} \to X_s$ in probability as $|s^n - s| \to 0$.

Thus it is reasonable to introduce the following class of K-parameter stochastic processes.

Definition 2.1. A family $\{X_s \colon s \in K\}$ of random variables on $\mathbb{R}^d$ is called a *K-parameter Lévy process in law* on $\mathbb{R}^d$ if it satisfies (2.6), (2.8), (2.9), and (2.10).

Even in the $\mathbb{R}_+$-parameter case, there is a process which is not a Lévy process in law but corresponds to a convolution semigroup by (2.5) (see [23] E 12.15, E 18.18). However, in seeking a class of processes corresponding to convolution semigroups, we cannot find any meaningful one other than the class of K-parameter Lévy processes in law defined above. In the case $K = \mathbb{R}_+^N$, K-parameter Lévy processes in law in the sense of Definition 2.1 were introduced by Barndorff-Nielsen, Pedersen, and Sato [2] in relation to multivariate subordination.

Now we face two problems. (1) Given a K-parameter convolution semigroup $\{\mu_s \colon s \in K\}$, can we find a K-parameter Lévy process in law $\{X_s \colon s \in K\}$ satisfying (2.5)? (2) In the case where we can find $\{X_s \colon s \in K\}$ of (1), is it unique in law? We have obtained negative answers to both problems in [21]. Thus we have to pose the problem to seek conditions under which (1) or (2) is answered in the affirmative. This is a difficult problem and we have only partial answers, which we describe below. We refer to [21] for proofs.

Definition 2.2. A K-parameter convolution semigroup $\{\mu_s \colon s \in K\}$ is said to be *generative* if there is a K-parameter Lévy process in law $\{X_s \colon s \in K\}$ associated with it, that is, satisfying (2.5).

Definition 2.3. A generative K-parameter convolution semigroup is said to be *unique-generative* if all K-parameter Lévy processes in law associated with it have the same system of finite-dimensional distributions. Otherwise it is said to be *multiple-generative.*

One might be tempted to call a process associated with the canonical $\mathbf{S}_d^+$-parameter convolution semigroup an $\mathbf{S}_d^+$-parameter Brownian motion. But such a process does not exist.

Theorem 2.4. *For any $d \geqslant 2$, the canonical $\mathbf{S}_d^+$-parameter convolution semigroup is non-generative.*

Another remarkable fact on the cone $\mathbf{S}_d^+$ is the following.

Theorem 2.5. *Let $d \geqslant 2$ and let $K = \mathbf{S}_d^+$. Let $\{\mu_s \colon s \in K\}$ be a K-parameter convolution semigroup such that $\int |x|^2 \mu_s(dx) < \infty$ and the covariance matrix v_s of μ_s satisfies $v_s \leqslant_K s$ for every $s \in K$. Then μ_s is Gaussian for every $s \in K$.*

In the case of $\mathbb{R}_+ = \mathbf{S}_1^+$, the condition $v_t \leqslant t$ is fulfilled by all $\mathbb{R}_+$-parameter convolution semigroups $\{\mu_t \colon t \in \mathbb{R}_+\}$ on $\mathbb{R}$ with the triplet (A, ν, γ) of μ_1 satisfying $A + \int_{\mathbb{R}} x^2 \nu(dx) \leqslant 1$.

Let K be a cone in $\mathbb{R}^M$. Let us denote the triplet of μ_s by (A_s, ν_s, γ_s).

Theorem 2.6. *If $\{\mu_s \colon s \in K\}$ is a K-parameter convolution semigroup on $\mathbb{R}^d$ which is purely non-Gaussian in the sense that $A_s = 0$, then $\{\mu_s \colon s \in K\}$ is generative.*

Theorem 2.7. *If $\{\mu_s \colon s \in K\}$ is a K-parameter convolution semigroup on $\mathbb{R}$ (that is, $d = 1$), then it is generative.*

Theorem 2.8. *If K has a strong basis, then, for any d, any K-parameter convolution semigroup $\{\mu_s \colon s \in K\}$ on $\mathbb{R}^d$ is generative.*

A distribution is called trivial if it is concentrated at a point. A convolution semigroup $\{\mu_s \colon s \in K\}$ is called trivial if each μ_s is trivial.

Theorem 2.9. *Let K possess a strong basis $\{e^1, \ldots, e^N\}$ and let $\{\mu_s \colon s \in K\}$ be a K-parameter convolution semigroup on $\mathbb{R}^d$. If $A_{e^j}(\mathbb{R}^d) \cap A_{e^k}(\mathbb{R}^d) \neq \{0\}$ for some $j \neq k$ or if ν_{e^j} and ν_{e^k} are not mutually singular for some $j \neq k$, then $\{\mu_s\}$ is multiple-generative. In particular, if $\{\mu_s\}$ is non-trivial and if $\mu_{e^1} = \cdots = \mu_{e^N}$, then $\{\mu_s\}$ is multiple-generative.*

Theorem 2.10. *Let K possess a strong basis $\{e^1, \ldots, e^N\}$. Let $L_1, \ldots, L_N \in \mathcal{B}(\mathbb{R}^d)$ be additive subgroups of $\mathbb{R}^d$ such that $L_j \cap L_k = \{0\}$ for all $j \neq k$. If $\{\mu_s \colon s \in K\}$ is a K-parameter convolution semigroup satisfying $\mu_{te^j}(L_j) = 1$ for all j and t, then $\{\mu_s\}$ is unique-generative.*

Theorem 2.11. *Let K possess a strong basis. If $\{\mu_s\colon s \in K\}$ is a unique-generative K-parameter convolution semigroup on $\mathbb{R}^d$, then all finite-dimensional distributions of the associated K-parameter Lévy process in law are infinitely divisible. But there is a multiple-generative K-parameter convolution semigroup $\{\mu_s\colon s \in K\}$ on $\mathbb{R}^d$ such that some finite-dimensional distributions of some K-parameter Lévy process in law associated with $\{\mu_s\}$ are not infinitely divisible.*

Among these results, Theorem 2.8 and the last statement of Theorem 2.9 are easy to prove (see Example 2.12).

Example 2.12. Let K be a cone with a strong basis $\{e^1, \ldots, e^N\}$. If $\{V_t^j\colon t \in \mathbb{R}_+\}$, $j = 1, \ldots, n$, are independent Lévy processes in law on $\mathbb{R}^d$ and if $u^j = u_1^j e^1 + \cdots + u_N^j e^N \in K$ for $j = 1, \ldots, n$, then $\{X_s\colon s \in K\}$ defined by

$$X_s = \sum_{j=1}^{n} V^j_{u_1^j s_1 + \cdots + u_N^j s_N} \quad \text{for } s = s_1 e^1 + \cdots + s_N e^N \tag{2.11}$$

is a K-parameter Lévy process in law.

If we are given a K-parameter convolution semigroup $\{\mu_s\colon s \in K\}$ on $\mathbb{R}^d$, then, choosing $n = N$, $u_k^j = 0$ or 1 according as $k \neq j$ or $k = j$, and $\mathcal{L}(V_1^j) = \mu_{e^j}$ for $j = 1, \ldots, N$, we get a K-parameter Lévy process in law associated with $\{\mu_s\}$.

If we are given a K-parameter convolution semigroup $\{\mu_s\colon s \in K\}$ on $\mathbb{R}^d$ satisfying $\mu_{e^1} = \cdots = \mu_{e^N}$, then another construction of an associated K-parameter Lévy process in law is to choose $n = 1$ and $u^1 = e^1 + \cdots + e^N$. The system of finite-dimensional distributions of this K-parameter Lévy process in law is different from that of the former construction unless $\{\mu_s\}$ is trivial.

Hirsch [11] and Koshnevisan, Xiao, and Zhong [14] study potential theory and local times of the processes of type (2.11).

The process defined by (2.11) has an additional property that it has homogeneous independent increments along *any* straight line intersected with K. Inoue [12] gives a more general construction of similar K-parameter processes.

Example 2.13. Let K be a cone with a strong basis $\{e^1, \ldots, e^N\}$. For each $j = 1, \ldots, N$, let $\{V_t^j\colon t \in \mathbb{R}_+\}$ be a Lévy process in law on $\mathbb{R}^{d_j}$. Assume that they are independent. Define a process X_s by

$$X_s = (V^1_{s_1}, \ldots, V^N_{s_N})^\top \quad \text{for } s = s_1 e^1 + \cdots + s_N e^N. \tag{2.12}$$

Then $\{X_s\colon s\in K\}$ is a K-parameter Lévy process in law on $\mathbb{R}^d$, where $d=d_1+\cdots+d_N$. This is a special case of multi-parameter Markov processes whose potential theory is studied by Dynkin [7], Evans [9], Fitzsimmons and Salisbury [10], and others.

Remark 2.14. Let K_1 and K_2 be cones in $\mathbb{R}^{M_1}$ and $\mathbb{R}^{M_2}$, respectively. Let us give an analogue to Theorem 1.7 for cone-parameter Lévy processes in law. Hence let $\{Z_s\colon s\in K_1\}$ be a K_1-parameter Lévy process in law on $\mathbb{R}^{M_2}$ satisfying that $Z_s\in K_2$ a. s. for $s\in K_1$. Let $\{X_u\colon u\in K_2\}$ be a K_2-parameter Lévy process in law on $\mathbb{R}^d$ which is independent of $\{Z_s\colon s\in K_1\}$. Define $\{Y_s\colon s\in K_1\}$ by $Y_s=X_{Z'_s}$, where $Z'_s=Z_s 1_{K_2}(Z_s)$. Note that $Z'_s=Z_s$ a. s. In order to ensure that Y_s is measurable we assume that both $\{Z_s\colon s\in K_1\}$ and $\{X_u\colon u\in K_2\}$ are measurable processes. (That is, $(s,\omega)\mapsto Z_s(\omega)$ and $(u,\omega)\mapsto X_u(\omega)$ are both measurable mappings.) This is essentially no restriction since any cone-parameter Lévy process in law has a measurable modification; see [21]. By repeating the proof of Theorem 3.3 of [2] it follows that $\{Y_s\colon s\in K_1\}$ is a K_1-parameter Lévy process in law on $\mathbb{R}^d$. That is, the class of cone-parameter Lévy processes in law is closed under subordination. If we denote $\mu_u=\mathcal{L}(X_u)$, $\rho_s=\mathcal{L}(Z_s)$ and $\sigma_s=\mathcal{L}(Y_s)$, then we have the relation (1.11) between $\{\mu_u\colon u\in K_2\}$, $\{\rho_s\colon s\in K_1\}$ and $\{\sigma_s\colon s\in K_1\}$.

Example 2.15. Consider the case $K=\mathbb{R}_+^N$. Let us show that Brownian sheets studied by Orey and Pruitt [19], Talagrand [27], Khoshnevisan and Shi [13], and many others, and the so-called multi-parameter Lévy processes for $N\geqslant 2$ studied by Ehm [8] (in the strictly stable case), Vares [28], and Lagaize [15] (both in the case $N=2$) are not K-parameter Lévy processes in law in our sense.

We consider the case $N=2$, but the case of general N is similar. The processes in the papers mentioned above are in the following category. Let $\{X_s\colon s\in\mathbb{R}_+^2\}$ be a family of random variables on $\mathbb{R}^d$. Now and then we write X_{s_1,s_2} instead of X_s when $s=(s_1,s_2)^\top$. For $s=(s_1,s_2)^\top$ and $u=(u_1,u_2)^\top$ in $K=\mathbb{R}_+^2$ with $s\leqslant_K u$, call $B=(s_1,u_1]\times(s_2,u_2]$ a rectangle in $\mathbb{R}_+^2$ and set

$$X(B)=X_{u_1,u_2}-X_{s_1,u_2}-X_{u_1,s_2}+X_{s_1,s_2},$$

If $B_1,\dots,B_n$ are disjoint rectangles in $\mathbb{R}_+^2$ and $B=\bigcup_{j=1}^n B_j$, then set $X(B)=\sum_{j=1}^n X(B_j)$. Assume the following properties.

(a) If $n\geqslant 2$ and $B_1,\dots,B_n$ are disjoint rectangles, then $X(B_1),\dots,X(B_n)$ are independent.

(b) If B is a rectangle and $s\in\mathbb{R}_+^2$, then $X(B)\overset{\mathrm{d}}{=}X(B+s)$.

(c) $X_{s_1,0} = X_{0,s_2} = 0$ a.s. for $s_1, s_2 \in \mathbb{R}_+$.

(d) $X_{s'} \to X_s$ in probability as $|s' - s| \to 0$ in $\mathbb{R}_+^2$.

We call a process satisfying (a)–(d) a *Lévy sheet* on $\mathbb{R}^d$. It follows that if $\{X_s \colon s \in \mathbb{R}_+^2\}$ is a Lévy sheet then $\mathcal{L}(X_s)$ is infinitely divisible and that, denoting $\mu = \mathcal{L}(X_{1,1})$, we have

$$E\left[e^{i\langle z, X_s\rangle}\right] = \widehat{\mu}(z)^{s_1 s_2} \quad \text{for } s = (s_1, s_2)^\top \in \mathbb{R}_+^2. \tag{2.13}$$

Thus we have

$$E\left[e^{i\langle z, X(B)\rangle}\right] = \widehat{\mu}(z)^{\mathrm{Leb}\, B}, \tag{2.14}$$

for any B, union of disjoint rectangles, where $\mathrm{Leb}\, B$ is the Lebesgue measure of B. Let $\mu_s = \mathcal{L}(X_s)$. Now we assume that $\mu \neq \delta_0$ and show the following:

(i) $\{X_s \colon s \in \mathbb{R}_+^2\}$ satisfies (2.8);

(ii) if $s^1, \ldots, s^4 \in \mathbb{R}_+^2$ satisfy $s^2 - s^1 = s^4 - s^3 \in \mathbb{R}_+^2 \setminus \{0\}$ and $s^3 - s^1 \in \mathbb{R}_+^2 \setminus \{0\}$, then $\mathcal{L}(X_{s^2} - X_{s^1}) \neq \mathcal{L}(X_{s^4} - X_{s^3})$;

(iii) if $s = (s_1, s_2)^\top$ and $u = (u_1, u_2)^\top$ in $\mathbb{R}_+^2$ satisfy $s_1 u_2 \neq 0$ or $s_2 u_1 \neq 0$, then $\mu_s * \mu_u \neq \mu_{s+u}$.

Indeed, to show (i), let

$$B_j = ((0, s_1^{j+1}] \times (0, s_2^{j+1}]) \setminus ((0, s_1^j] \times (0, s_2^j]).$$

Then $X_{s^{j+1}} - X_{s^j} = X(B_j)$. Since $B_1, \ldots, B_n$ are disjoint, $X(B_1), \ldots, X(B_n)$ are independent.

To show (ii), let B_1 and B_3 be as above, note that $\mathrm{Leb}\, B_1 < \mathrm{Leb}\, B_3$, and use (2.14).

To show (iii), note that, by (2.13), the characteristic function of $\mu_s * \mu_u$ is $\widehat{\mu}(z)^{s_1 s_2 + u_1 u_2}$ while that of μ_{s+u} is $\widehat{\mu}(z)^{(s_1+u_1)(s_2+u_2)}$.

For general N, a class of $\mathbb{R}_+^N$-parameter stochastic processes satisfying (a) and (d) is studied by Adler et al. [1].

In Remark 2.14 we showed that the class of cone-parameter Lévy processes in law is closed under subordination. We conclude the present example by showing that this property is not shared by Lévy sheets. Let $\{X_{u_1,u_2} \colon (u_1, u_2)^\top \in \mathbb{R}_+^2\}$ be a Lévy sheet on $\mathbb{R}^d$ and μ denote the infinitely divisible distribution on $\mathbb{R}^d$ which determines the law of X_{u_1,u_2} as in (2.13). Let $\{Z_{s_1,s_2} \colon (s_1, s_2)^\top \in \mathbb{R}_+^2\}$ be a Lévy sheet such that $Z_{s_1,s_2} \in \mathbb{R}_+^2$ almost surely for all $(s_1, s_2)^\top \in \mathbb{R}_+^2$. Assume that these two Lévy sheets are independent and define $Y_{s_1,s_2} = X_{Z_{s_1,s_2}}$. We give two examples where $\{Y_{s_1,s_2} \colon (s_1, s_2)^\top \in \mathbb{R}_+^2\}$ is not a Lévy sheet. Firstly, if we take $Z_{s_1,s_2} = (s_1 s_2, s_1 s_2)$

for $(s_1, s_2)^\top \in \mathbb{R}_+^2$ then, noting that $\{Z_{s_1,s_2} \colon (s_1, s_2)^\top \in \mathbb{R}_+^2\}$ is indeed a Lévy sheet on $\mathbb{R}_+^2$, we have $Y_{s_1,s_2} = X_{s_1 s_2, s_1, s_2}$ and Y_{s_1,s_2} has characteristic function $\widehat{\mu}(z)^{(s_1 s_2)^2}$ for $(s_1, s_2)^\top \in \mathbb{R}_+^2$. If $\mu \neq \delta_0$ this is incompatible with the structure of the characteristic function of a Lévy sheet in (2.13). Secondly we show that Y_{s_1,s_2} need not even be infinitely divisible. Let $X_{u_1,u_2} = u_1 u_2$, which is a Lévy sheet on $\mathbb{R}_+$. Then $Y_{s_1,s_2} = Z^1_{s_1,s_2} Z^2_{s_1,s_2}$, where $Z^i_{s_1,s_2}$ denotes the ith coordinate of Z_{s_1,s_2} for $i = 1, 2$. Hence, we just have to construct Z_{s_1,s_2} such that the product of the coordinates is not infinitely divisible and we conclude by giving one such construction. Let ν_1 and ν_2 denote infinitely divisible distributions on $\mathbb{R}_+$. Since the product measure $\nu_1 \times \nu_2$ is infinitely divisible on $\mathbb{R}_+^2$ there exists a Lévy sheet $\{Z_{s_1,s_2} \colon (s_1, s_2)^\top \in \mathbb{R}_+^2\}$ with

$$Ee^{i\langle z, Z_{s_1,s_2}\rangle} = \widehat{\nu}_1(z_1)^{s_1 s_2} \widehat{\nu}_2(z_2)^{s_1 s_2} \quad \text{for } z = (z_1, z_2)^\top \in \mathbb{R}^2 \text{ and } (s_1, s_2)^\top \in \mathbb{R}_+^2.$$

Thus, $Z^1_{s_1,s_2}$ and $Z^2_{s_1,s_2}$ are independent and $\mathcal{L}(Z^i_{s_1,s_2}) = \nu_i^{s_1 s_2}$ for $i = 1, 2$. If e.g. both ν_1 and ν_2 are the Poisson distribution with mean 1 then $Z^1_{s_1,s_2}$ and $Z^2_{s_1,s_2}$ have a Poisson distribution with mean $s_1 s_2$. As shown in Rohatgi et al. [22], the product of two such variables is not infinitely divisible for any $s = (s_1, s_2)^\top$ with $s_1 s_2 \neq 0$. More generally, if ν_1 and ν_2 are infinitely divisible distributions on $\mathbb{R}_+$ such that the product $V_1 V_2$ is not infinitely divisible for independent random variables V_1, V_2 with $\mathcal{L}(V_i) = \nu_i$, then the same construction gives Y_{s_1,s_2} which is not infinitely divisible for $s_1 s_2 = 1$. Such a pair ν_1, ν_2 was first given by Shanbhag et al. [25].

Example 2.16. Lévy [16] defined and studied an $\mathbb{R}^M$-parameter Brownian motion and the papers of Chentsov [5] and McKean [17] followed. Let $\{X_s \colon s \in \mathbb{R}^M\}$ be Lévy's $\mathbb{R}^M$-parameter Brownian motion. It is characterized in law by the properties that X_s is $\mathbb{R}$-valued and any finite-dimensional distribution is Gaussian and that $E[X_s] = 0$ and

$$E[X_{s^1} X_{s^2}] = \tfrac{1}{2}(|s^1| + |s^2| - |s^2 - s^1|) \quad \text{for } s^1, s^2 \in \mathbb{R}^M.$$

Hence $\mathcal{L}(X_s) = N(0, |s|)$ and $\mathcal{L}(X_{s^2} - X_{s^1}) = N(0, |s^2 - s^1|)$. Thus, for any cone K in $\mathbb{R}^M$, the restriction $\{X_s \colon s \in K\}$ to K has the property (2.9). However, the property (2.8) is far from valid. We have

$$E[(X_{s^2} - X_{s^1})(X_{s^4} - X_{s^3})] = \tfrac{1}{2}(|s^3 - s^2| + |s^4 - s^1| - |s^4 - s^2| - |s^3 - s^1|)$$

for all $s^1, \ldots, s^4 \in \mathbb{R}^M$. Hence $X_{s^2} - X_{s^1}$ and $X_{s^4} - X_{s^3}$ are independent if and only if

$$|s^3 - s^2| + |s^4 - s^1| = |s^4 - s^2| + |s^3 - s^1|.$$

For instance (letting $s^2 = s^3$) $X_{s^2} - X_{s^1}$ and $X_{s^4} - X_{s^2}$ are independent if and only if s^1, s^2, s^4 are on a straight line in this order. (See [16] for a geometric consideration of the independence condition.) Now we see that, for any N-dimensional cone K in $\mathbb{R}^M$ with $M \geqslant N \geqslant 2$, the restriction $\{X_s \colon s \in K\}$ of $\{X_s \colon s \in \mathbb{R}^M\}$ to K does not satisfy (2.8).

For any s^1 and s^2 with $|s^2 - s^1| = 1$, the restriction $\{X_{s^1+t(s^2-s^1)} \colon t \in \mathbb{R}_+\}$ is the usual ($\mathbb{R}_+$-parameter) Brownian motion (in law). Mori [18] characterized $\mathbb{R}^M$-parameter processes having the independent increment property along straight lines in the purely non-Gaussian setting.

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