DEFORMATION QUANTIZATION AND GEOMETRIC QUANTIZATION OF ABELIAN MODULI SPACES.

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ABSTRACT. The Berezin-Toeplitz deformation quantization of an abelian variety is explicitly computed by the use of Theta-functions. An explicit $SL(2n,\mathbb{Z})$-equivariant complex structure dependent equivalence E between the constant Moyal-Weyl product and this family of deformations is given. This equivalence is seen to be convergent on the dense subspace spanned by the pure phase functions. The Toeplitz operators associated to the equivalence E applied to a pure phase function produces a covariant constant section of the endomorphism bundle of the vector bundle of Theta-functions (for each level) over the moduli space of abelian varieties.

Applying this to any holonomy function on the symplectic torus one obtains as the moduli space of $U(1)$-connections on a surface, we provide an explicit geometric construction of the abelian TQFT-operator associated to a simple closed curve on the surface. Using these TQFT-operators we prove an analog of asymptotic faithfulness in this abelian case. Namely that the intersection of the kernels for the quantum representations is the Toreilli subgroup in this abelian case.

Furthermore, we relate this construction to the deformation quantization of the moduli spaces of flat connections constructed in [AMR1] and [AMR2]. In particular we prove that this topologically defined $*$-product in this abelian case is the Moyal-Weyl product. Finally we combine all of this to give a geometric construction of the abelian TQFT operator associated to any link in the cylinder over the surface and we show the glueing axiom for these operators.

1. INTRODUCTION

A very concrete link between geometric quantization and deformation quantization is provided by the Berezin-Toeplitz deformation quantization of a compact Kähler manifold. We are here in particular referring to the constructions of Bordeman, Meinrenken and Schlichenmaier [BMS] and Schlichenmaier [Sch]. Let us describe the basics of their constructions.

Let (M,ω) be a prequantizable compact symplectic manifold, i.e. there exist a Hermitian line bundle L with a connection whose curvature is the symplectic form. Suppose further that we have a complex structure I on M, which is compatible with ω, such that $M_I = (M, I, \omega)$ is a Kähler manifold.

The Berezin-Toeplitz deformation quantization is then obtained as follows. For any positive integer k consider the finite dimensional subspace $H^0(M_I, L^k)$ of holomorphic sections of L^k inside the Hilbert space of all L_2-sections. The Toeplitz

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operator $T_f^{(k)}$ of a smooth function $f \in C^\infty(M)$ is the operator on $H^0(M_I, L^k)$ defined as the composite of the multiplication with f and then the orthogonal projection onto $H^0(M_I, L^k)$. For a pair of smooth functions $f_1, f_2 \in C^\infty(M)$, one considers the asymptotic expansion in $1/k$ of the product $T_{f_1}^{(k)} T_{f_2}^{(k)}$ in terms of Toeplitz operators, and one finds that

$$T_{f_1}^{(k)} T_{f_2}^{(k)} \sim \sum_{l=0}^{\infty} T_{c_l(f_1,f_2)}^{(k)} k^{-l},$$

where $c_l(f_1, f_2) \in C^\infty(M)$ are uniquely determined (see Theorem 2 due to Schlichenmaier for the precise meaning of $\sim$) and gives the Berezin-Toeplitz deformation quantization

$$f \star_I g = \sum_{l=0}^{\infty} (-1)^l c_l(f,g) h^l,$$

of M_I.

In this paper we study the case where M_I is any principal polarized abelian variety. So let $M = V/\Lambda$, where V is a real vector space with a symplectic form ω, and Λ is a discrete lattice in V of maximal rank such that ω is integral and unimodular when restricted to Λ. Let now $\mathcal{C}$ be the space of complex structures on V, which are compatible with ω. Then for any $I \in \mathcal{C}$, $M_I = (M, I, \omega)$ is an abelian variety.

We compute in this paper (see section 4) the Berezin-Toeplitz $*$-product explicitly:

On $M = V/\Lambda$ we have the complex structure independent Moyal-Weyl $*$-product $*$ (as discussed in section 2). Now consider the formal transform

$$E_I = e^{-\frac{h}{2}\Delta_I} : C^\infty(M) \to C^\infty(M)$$

where Δ_I is the Laplace operator on M_I. We then have that

$$E_I^{-1}(E_I(f) \star_I E_I(g)) = f * g.$$

We prove this relation as follows: For any element in $\lambda \in \Lambda$, we get a pure phase function $F_\lambda \in C^\infty(M)$ and we consider the Toeplitz operators $T_{F_\lambda}^{(k)}$. By an explicit computation, we get a formula for the matrix coefficients of $T_{F_\lambda}^{(k)}$ with respect to the Theta-function basis of $H^0(M_I, L^k)$ (see formula (1)). Using this expression we can explicitly compute products of these Toeplitz operators and verify the above relation between the two products.

In fact, the vector spaces $H^0(M_I, L^k)$ form a vector bundle, say $H^{(k)}$, over $\mathcal{C}$, and there is a natural flat connection in this vector bundle. It is characterized by the fact that the Theta-function basis is covariant constant with respect to this connection. By the above mentioned computation one observes that the Toeplitz operators $T_{F_\lambda}^{(k)}$ are **not** covariant constant sections of $\operatorname{End}(H^{(k)})$. However, we observe that $E_I(F_\lambda) \in C^\infty(M)[[h]]$ is convergent for $h = 1/k$ and that $T_{E_I(F_\lambda)(1/k)}^{(k)}$ is covariant constant (see Remark 1). Let us now discuss how we apply this to Abelian gauge theory.

Let Σ be a closed oriented surface of genus g. Let M be the moduli space of flat $U(1)$-connections on Σ. Then

$$M = \operatorname{Hom}(\pi_1(\Sigma), U(1)) = H^1(\Sigma, \mathbb{R})/H^1(\Sigma, \mathbb{Z}).$$

There is the usual symplectic structure ω on $H^1(\Sigma, \mathbb{R})$ which is of course integral and unimodular over the lattice $H^1(\Sigma, \mathbb{Z})$. The mapping class group Γ of Σ acts on M via the induced homomorphims

$$\rho : \Gamma \to \mathrm{Aut}(H^1(\Sigma, \mathbb{Z}), \omega).$$

In [AMR1] and [AMR2] we constructed in collaboration with Mattes and Reshetikhin a $*$-product on the moduli space M^G of flat G-connections on the surface $\Sigma' = \Sigma - \{\mathrm{pt}\}$, where G is either $GL(m, \mathbb{C})$ or $SL(m, \mathbb{C})$. This $*$-product is constructed using a universal Vassiliev invariant for links in $\Sigma' \times [0,1]$ and the product on links in this manifold.

Notice that we in the abelian case $G = GL(1, \mathbb{C}) = \mathbb{C}^*$ have $M \subset M^G$. We argue in section 5 that this $*$-product on M^G for $G = \mathbb{C}^*$ restricts to a $*$-product on M, and we will show that the resulting $*$-product is the Moyal-Weyl product.

In $2+1$-dimensional Chern-Simons theory, the 2-dimensional part of the theory is a modular functor, which is a functor from the category of compact smooth oriented surfaces to the category of finite dimensional complex vector spaces, which satisfies certain properties. In the gauge-theoretic construction of this functor one first fixes a compact Lie group K and an invariant non-degenerate inner product on its Lie algebra. The functor then associates to a closed oriented surface the finite dimensional vector space one obtains by applying geometric quantization to the moduli space of flat K-connections on the surface (See e.g. [W1] and [At1]). - In the abelian case $K = U(1)$ at hand this means concretely the following. By applying the geometric quantization dicussed above to the abelian moduli space M, we get the vector bundle $H^{(k)}$ over the space of complex structures $\mathcal{C}$ on $H^1(\Sigma, \mathbb{R})$. This bundle has a flat connection, and an action of $\mathrm{Aut}(H^1(\Sigma, \mathbb{Z}), \omega)$, which preserves the flat connection. In this case the modular functor is defined by associating to Σ, the vector space $Z_k(\Sigma)$ consisting of covariant constant sections of $H^{(k)}$ over $\mathcal{C}$. So through the representation ρ, we get a representation ρ_k of the mapping class group Γ of Σ on $Z_k(\Sigma)$. In the non-abelian case the situation was actually developed generalizing from this abelian case, and one gets a (projective) flat vector bundle over Teichmüller space of Σ (see [ADW] and [H] and [vGdJ]). By restricting to the embedded copy of Teichmüller space of Σ in $\mathcal{C}$, we also get exactly this situation in the abelian case. However, there seems at present no analog in the non-abelian case of a construction of a (projective) flat vector bundle over $\mathcal{C}$.

In 2+1-dimensional Chern-Simons theory one also has the following TQFT setup. Suppose Y is a compact oriented 3-manifold such that $\partial Y = (-\Sigma_1) \cup \Sigma_2$, where Σ_1 and Σ_2 are closed oriented surfaces and $-\Sigma_1$ means Σ_1 with the orientation reversed. Assume further L is a link inside $Y \setminus \partial Y$. Then the TQFT-axioms states that there should be a linear morphism $Z_k(Y, L) : Z_k(\Sigma_1) \to Z_k(\Sigma_2)$, which satisfies that glueing along boundary components goes to the corresponding composition of linear maps.

In section 6 we give a geometric construction of these operators in the case where $Y = \Sigma \times [0,1]$. For a simple closed curve γ on Σ, this operator

$$Z_k(\gamma) = Z_k(\Sigma \times [0,1], \gamma) \in \mathrm{Hom}(Z_k(\Sigma))$$

is constructed as follows. We consider the holonomy function $F_\gamma \in C^\infty(M)$. This is a pure phase function, hence the series $E_I(F_\gamma) \in C^\infty(M)[[h]]$ is convergent for $h = 1/k$ for all $I \in \mathcal{C}$. According to above discussion $T^{(k)}_{E_I(F_\gamma)(1/k)}$ gives a covariant constant section of $\mathrm{End}(H^{(k)})$ as I sweeps through $\mathcal{C}$. We define $Z_k(\gamma)$ to be this section.

In section 6, we use these operators to give a TQFT proof of the following well known classical result from the theory of Theta-functions

$$\bigcap_{k=1}^{\infty} \ker \rho_k = \ker \rho,$$

i.e. the action of the symplectomorphism group of the lattice is asymptotic faithful on Theta-functions of all levels. In [A1] we have extended this result to the non-abelian case, where one gets the much stronger result that the action of the mapping class group is asymptotic faithful (see Theorem 1 in [A1] for the precise statement).

Finally, we combine all of the above to give a geometric construction of

$$Z_k(\Sigma \times [0,1], L) \in \mathrm{Hom}(Z_k(\Sigma))$$

for any link L as follows. We apply the universal Vassiliev invariant constructed in [AMR2] to the link L. The result in this abelian case is an infinite series in h with coefficients in $H^1(\Sigma, \mathbb{Z})$. By taking the associated holonomy functions, we get $F_L \in C^\infty(M)[[h]]$. One sees that $E_I(F_L) \in C^\infty(M)[[h]]$ is convergent for $h = 1/k$ and we define $Z_k(\Sigma \times [0,1], L) \in \mathrm{Hom}(Z_k(\Sigma))$ to be the covariant constant section $T^{(k)}_{E_I(F_L)(1/k)}$. Combining all of the above relations, we get the needed glueing relation

$$Z_k(\Sigma \times [0,1], L_1.L_2) = Z_k(\Sigma \times [0,1], L_1) \circ Z_k(\Sigma \times [0,1], L_2).$$

We consider it a rather interesting problem to generalize all the constructions presented in this paper to the non-abelian case. In [A1] we have taken a number of steps in this direction.

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2. Deformation quantization

In this section we will very briefly review the basic setup in deformation quantization of Poisson manifolds. We refer the read to [daSW] and the references in there for a more detailed discussion.

Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold. That is we have a bilinear anti-symmetric pairing $\{\cdot,\cdot\} : C^\infty(M) \otimes C^\infty(M) \to C^\infty(M)$, which satisfies the Leibniz rule in each variable. Let $C^\infty_h(M) = C^\infty(M)[[h]]$ and $\mathbb{C}_h = \mathbb{C}[[h]]$. The notion of a deformation quantization of a Poisson manifold was introduced in [BFFLS].

Definition 1. *A (formal) deformation quantization of (or $*$-product on) $(M, \{\cdot,\cdot\})$ is an associative $\mathbb{C}_h$-algebra structure $*$ on $C^\infty_h(M)$ such that*

$$f * g = fg \ mod\ h$$
$$f * g - g * f = h\{f,g\} \ mod\ h^2$$

for all $f, g \in C^\infty(M)$.

We will further only study $*$-products such that $1 \in C^\infty(M)$ is also a $*$-unit. Any deformation quantization induces bilinear maps

$$C_r : C^\infty(M) \otimes C^\infty(M) \to C^\infty(M),$$

which determines the product completely

$$f * g = \sum_r C_r(f,g)h^r, \;\; f,g \in C^\infty(M).$$

We say that $*$ is differentiable if C_r are bidifferential operators.

Example. Consider $\mathbb{R}^{2n}$ with the standard symplectic structure given in the standard coordinates (x_i, y_i) by

$$\omega = \sum_i dx_i \wedge dy_i.$$

Consider the operator $\hat{P} : C^\infty(\mathbb{R}^{2n} \times \mathbb{R}^{2n}) \to C^\infty(\mathbb{R}^{2n} \times \mathbb{R}^{2n})$ given in coordinates $(x_i', y_i', x_i'', y_i'')$ on $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ by

$$\hat{P} = \sum_i \left(\frac{\partial}{\partial x_i'} \frac{\partial}{\partial y_i''} - \frac{\partial}{\partial y_i'} \frac{\partial}{\partial x_i''} \right).$$

If we now denote the restriction map to the diagonal by $D : C^\infty(\mathbb{R}^{2n} \times \mathbb{R}^{2n}) \to C^\infty(\mathbb{R}^{2n})$ and the canonical map from $C^\infty(\mathbb{R}^{2n}) \otimes C^\infty(\mathbb{R}^{2n})$ to $C^\infty(\mathbb{R}^{2n} \times \mathbb{R}^{2n})$ by ι, then

$$\{f, g\} = D \circ \hat{P} \circ \iota(f \otimes g).$$

Extend ι, D and $\hat{P}$ to C_h^∞ by requiring $\mathbb{C}_h$-linearity. Now define the Moyal-Weyl product on $\mathbb{R}^{2n}$ by

$$f * g = D \circ \exp(\frac{h}{2}\hat{P}) \circ \iota(f \otimes g).$$

It is clear that this product induces a deformation quantization of the Poisson bracket of the standard structure on $\mathbb{R}^{2n}$.

We now consider the action of a maximal rank lattice Λ in $\mathbb{R}^{2n}$ on $C^\infty(\mathbb{R}^{2n})$ induced by $\mathbb{R}^{2n}$ acting symplectically on it self by translations. We observe that $D, \hat{P}$ and ι are equivariant with respect to the action of Λ. Hence the Moyal-Weyl product on $\mathbb{R}^{2n}$ induces a deformation quantization of the torus $M = \mathbb{R}^{2n}/\Lambda$. We also denote this product on $C_h^\infty(M)$ by $*$.

For pure phases, we can explicitly compute this $*$-product. We may without loss of generality assume that $\Lambda \cong \mathbb{Z}^{2n} \subset \mathbb{R}^{2n}$. For each $\lambda \in \Lambda$ consider the functions $F_\lambda \in C^\infty(M)$ given by

$$F_\lambda(v) = \exp(2\pi i \lambda \cdot v)$$

where $v \in \mathbb{R}^{2n}$ and $\cdot$ denote the usual euclidian inner product.

Lemma 1. *For all* $\lambda, \lambda' \in \Lambda$ *we have that*

$$F_\lambda * F_{\lambda'} = \exp(\pi i h \omega(\lambda, \lambda')) F_{\lambda + \lambda'}.$$

Proof. Using the standard symplectic coordinates (x,y) on $\mathbb{R}^{2n}$ and induced coordinates (x', y', x'', y'') on $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$, we have that

$$\iota(F_\lambda \otimes F_{\lambda'})(x', y', x'', y'') = \exp(2\pi i(a \cdot x' + b \cdot y' + c \cdot x'' + d \cdot y'')),$$

where $\lambda = (a, b)$ and $\lambda' = (c, d)$. But then we see that

$$\hat{P}(\iota(F_\lambda \otimes F_{\lambda'})) = 2\pi i(a \cdot d - b \cdot c)\iota(F_\lambda \otimes F_{\lambda'})$$

and

$$\exp(\frac{h}{2}\hat{P})(\iota(F_\lambda \otimes F_{\lambda'})) = \exp(\pi i h(a \cdot d - b \cdot c))\iota(F_\lambda \otimes F_{\lambda'}).$$

By restriction to the diagonal we get the stated formula.

□

3. Berezin-Toeplitz deformation quantization.

Let (M, ω) be a compact symplectic manifold. Assume that (M, ω) is prequantizable, so that we can fix a Hermitian line bundle with a compatible connection $(L, \langle\cdot,\cdot\rangle, \nabla)$ such that the curvature of ∇ satisfies

$$F_\nabla = \frac{i}{2\pi}\omega.$$

Such a bundle is called a prequantum line bundle.

For each $k \in \mathbb{Z}$ we can consider the pre-Hilbert space

$$\mathcal{H}^{(k)} = C^\infty(M, L^k)$$

consisting of smooth sections of L^k. Integrating the inner product of two sections against the volume form associated to the symplectic form gives the pre-Hilbert space structure

$$(s_1, s_2) = \frac{1}{n!}\int_M \langle s_1, s_2\rangle \omega^n.$$

For each $f \in C^\infty(M)$ we consider the prequantum operator, namely the differential operator $P_f^{(k)} : C^\infty(M, L^k) \to C^\infty(M, L^k)$ given by

$$P_f^{(k)} = -\frac{1}{k}\nabla_{X_f}^{(k)} + if\cdot$$

where X_f is the Hamiltonian vector field associated to f.

Now assume that we have a complex structure I on M, which is compatible with ω. We can then consider the subspace $H_I^{(k)}$ of $\mathcal{H}^{(k)}$ consisting of holomorphic sections of L^k:

$$H_I^{(k)} = H^0(M_I, L^k).$$

By standard elliptic theory this is a finite dimensional subspace of $\mathcal{H}^{(k)}$ and we have the orthogonal projection $\pi^{(k)} : \mathcal{H}^{(k)} \to H_I^{(k)}$. From this projection we can construct the Toeplitz operator associated to any smooth function $f \in C^\infty(M)$, $T_f^{(k)} : \mathcal{H}^{(k)} \to H_I^{(k)}$, defined by

$$T_f^{(k)}(s) = \pi^{(k)}(fs)$$

for any element s in $\mathcal{H}^{(k)}$. We recall by Tuynman's theorem (see [Tuyn]) that if we compose the prequantum operator associated to f by the orthogonal projection, then it can be rewritten as a Toeplitz operator:

Theorem 1 (Tuynman). *For any $f \in C^\infty(M)$ we have that*

$$\pi^{(k)} \circ P_f^{(k)} = iT^{(k)}_{f - \frac{1}{2k}\Delta f}$$

as operators from $\mathcal{H}^{(k)}$ to $H_I^{(k)}$, where Δ is the Laplacian on (M, ω, I).

We shall interpret this theorem in the light of deformation quantization of a torus in remark 2 in section 4.

Let us now recall how one constructs a particular deformation quantization on a compact Kähler manifold following Schlichenmaier [Sch].

Theorem 2 (Schlichenmaier). *For any pair of smooth functions $f_1, f_2 \in C^\infty(M)$, we have an asymptotic expansion*

$$T^{(k)}_{f_1} T^{(k)}_{f_2} \sim \sum_{l=0}^{\infty} T^{(k)}_{c_l(f_1,f_2)} k^{-l},$$

where $c_l(f_1, f_2) \in C^\infty(M)$ are uniquely determined since $\sim$ means the following: For all $L \in \mathbb{Z}_+$ we have that

$$\|T^{(k)}_{f_1} T^{(k)}_{f_2} - \sum_{l=0}^{L} T^{(k)}_{c_l(f_1,f_2)} k^{-l}\| = O(k^{-(L+1)}).$$

Moreover, $c_0(f_1, f_2) = f_1 f_2$.

This theorem is proved in [Sch], where it is also proved that the formal generating series for the $c_l(f_1, f_2)$'s gives a formal deformation quantization[1] of the Poisson structure on M induced from ω.

Definition 2. *The Berezin-Toeplitz deformation quantization $\star_I$ of the compact Kähler manifold (M, ω, I) is*

$$f \star_I g = \sum_{l=0}^{\infty} (-1)^l c_l(f, g) h^l,$$

where $f, g \in C^\infty(M)$ and $c_l(f, g)$ are determined by Theorem 2.

In [KS], this Berezin-Toeplitz deformation quantization is identified in terms of Karabegov's classification of $*$-products with separation of variables on Kähler manifolds. Let $\star'_I$ be the unique $*$-product with separation of variables whose Karabegov form is

$$\tilde{\omega} = -\frac{1}{h}\omega + \rho$$

where ρ is the Ricci form, which is the curvature form of the Chern connection in the canonical bundle (see [K]). Let $B_I : C^\infty_h(M) \to C^\infty_h(M)$ be the formal Berezin Transform for $\star'_I$ (see section 2 in [KS]).

Theorem 3 (Karabegov & Schlichenmaier). *The Berezin-Toeplitz $*$-product $\star_I$ is related to $\star'_I$ via the formal Berezin-transform:*

$$f \star_I g = B_I^{-1}(B_I(f) \star'_I B_I(g)).$$

We will also need the following theorem due to Bordemann, Meinrenken and Schlichenmaier (see [BMS]).

Theorem 4 (Bordemann, Meinrenken and Schlichenmaier). *For any $f \in C^\infty(M)$ we have that*

$$\lim_{k\to\infty} \|T^{(k)}_f\| = \sup_{x\in M} |f(x)|.$$

Since the association of the sequence of Toeplitz operators T^k_f, $k \in \mathbb{Z}_+$ is linear in f, we see from this theorem, that this association is faithful.

[1] We have the opposite sign-convention on the curvature, which means our c_l are $(-1)^l c_l$ in [Sch].

4. Berezin-Toeplitz deformation quantization of abelian varieties

Let us now consider the case where M, the symplectic manifold, is a torus, i.e.

$$M = V/\Lambda,$$

where V is a real vector space equipped with a symplectic structure ω, and Λ is a discrete lattice in V of maximal rank such that ω is integral and unimodular when restricted to Λ. Then there exists a symplectic basis $(\lambda_1, \dots, \lambda_{2n})$ over the integers for Λ (see e.g. [GH] p. 304). Let $(x_1, \dots, x_n, y_1, \dots, y_n)$ be the dual coordinates on V. Then

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Let A be the automorphism group of (Λ, ω). Then A injects into the symplectomorphisms of (M, ω). Using the basis $(\lambda_1, \dots, \lambda_{2n})$, we get an identification $A \cong Sp(2n, \mathbb{Z})$.

Let now $\mathcal{C}$ be the space of complex structures on V, which are compatible with ω, i.e. $\mathcal{C}$ consists of the symplectomorphisms $I : V \to V$ such that the symmetric form $\omega(\cdot, I\cdot)$ is a positive definite inner product on V. For an $I \in \mathcal{C}$ the triple $M_I = (M, \omega, I)$ is a principal polarized abelian variety. Notice that A acts on $\mathcal{C}$.

Using the basis, we can identify $\mathcal{C}$ with the Siegel generalized upper half space

$$\mathbb{H} = \{Z \in M_{n,n}(\mathbb{C}) \mid Z = Z^t, \ \mathrm{Im}(Z) > 0\}.$$

For any $I \in \mathcal{C}$, we have that $(\lambda_1, \dots, \lambda_n)$ is basis over $\mathbb{C}$ for V with respect to I. Let $(z_1, \dots, z_n)$ be the dual complex coordinates on V relative to the basis $(\lambda_1, \dots, \lambda_n)$. The complex structure I determines and is determined by a unique $Z \in \mathbb{H}$ such that

$$z = x + Zy.$$

Since any $Z \in \mathbb{H}$ gives a positive complex structure, say $I(Z)$, compatible with the symplectic form, we have a bijective map $I : \mathbb{H} \to \mathcal{C}$, given by sending $Z \in \mathbb{H}$ to $I(Z)$. For a $Z \in \mathbb{H}$, we use the notation $X = \mathrm{Re}(Z)$ and $Y = \mathrm{Im}(Z)$.

Let us now for each $I \in \mathcal{C}$ explicitly construct a holomorphic prequantum line bundle $\mathcal{L}_I$ over M_I, by providing a lift of the Λ action on V to the trivial line bundle $\tilde{\mathcal{L}} = V \times \mathbb{C}$, such that the quotient is $\mathcal{L}_I$.

To this end we need a system of multipliers $e_\lambda \in C^\infty(V)$, which are non-where vanishing, holomorphic with respect to I and satisfying the following compatibility relations

$$e_{\lambda'}(v + \lambda) e_\lambda(v) = e_{\lambda'}(v) e_\lambda(v + \lambda') = e_{\lambda + \lambda'}(v)$$

for all $\lambda, \lambda' \in \Lambda$. The action of Λ on $\tilde{\mathcal{L}}$ is then given by

$$\lambda(v, z) = (v + \lambda, e_\lambda(v) z),$$

for all $\lambda \in \Lambda$ and $(v, z) \in \tilde{\mathcal{L}}$.

If we fix the multipliers for the basis $(\lambda_1, \dots, \lambda_{2n})$, then the compatibility equations uniquely determine the multipliers for all $\lambda \in \Lambda$. We fix the multipliers for $I(Z)$ as follows:

$$\begin{aligned} e_{\lambda_i}(z) &= 1, & i &= 1, \dots, n, \\ e_{\lambda_i}(z) &= e^{-2\pi i z_i - \pi i Z_{ii}}, & i &= n+1, \dots, 2n. \end{aligned}$$

Let us now introduce a Hermitian structure in $\mathcal{L}_I$. First we introduce the following function:

$$h(z) = e^{-2\pi y \cdot Y y}$$

where as above $z = x + Zy$. On $\tilde{\mathcal{L}}$ we consider the Hermitian structure $h\langle \cdot, \cdot \rangle_{\mathbb{C}}$, where $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ is the standard inner product on $\mathbb{C}$. Since

$$h(z+\lambda) = \frac{1}{|e_\lambda(z)|^2} h(z),$$

we see that this Hermitian structure is Λ-invariant and induces a Hermitian structure $\langle \cdot, \cdot \rangle$ on $\mathcal{L}_I$. By general theory, see e.g. section 2.6 in [GH], we have that the Chern connection of $(\mathcal{L}_I, \langle \cdot, \cdot \rangle)$ has curvature $\frac{i}{2\pi}\omega$.

The space of holomorphic sections of $\mathcal{L}_I^k$, $H^0(M_I, \mathcal{L}_I^k)$, has dimension k^n, and they give a vector bundle $H^{(k)}$ over $\mathcal{C}$, by letting $H_I^{(k)} = H^0(M_I, \mathcal{L}_I^k)$.

The L^2-inner product on $H^0(M_I, \mathcal{L}_I^k)$ is given by

$$(s_1, s_2) = \int_M s_1(z) \overline{s_2(z)} e^{-2\pi k y \cdot Y y} dx dy$$

for $s_1, s_2 \in H^0(M_I, \mathcal{L}_I^k)$.

We can give an explicit basis for this space in terms of the classical Theta-functions of level k:

$$\Theta_{\alpha,k}(Z,z) = \sum_{l \in \mathbb{Z}^n} e^{\pi i k (l+\alpha)\cdot Z(l+\alpha)} e^{2\pi i k (l+\alpha)\cdot z},$$

where $\alpha \in \frac{1}{k}\mathbb{Z}^n/\mathbb{Z}^n$.

These Theta-functions satisfy the following heat equation

$$\frac{\partial \Theta_{\alpha,k}}{\partial Z_{ij}} = -\frac{1}{4\pi i k} \frac{\partial^2 \Theta_{\alpha,k}}{\partial z_i \partial z_j}.$$

The geometric significance of this equation is as follows. Let us define a connection $\tilde{\mathbb{D}}$ in the trivial $C^\infty(\mathbb{C}^n)$-bundle over $\mathbb{H}$, by the following assignment

$$\tilde{\mathbb{D}}_{\frac{\partial}{\partial Z_{ij}}} = \frac{\partial}{\partial Z_{ij}} + \frac{1}{4\pi i k} \frac{\partial^2}{\partial z_i \partial z_j}.$$

Using the coordinates $z = x + Zy$ over the point $Z \in \mathbb{H}$ to identify $H^0(M_{I(Z)}, \mathcal{L}_I^k)$ with a subspace of $C^\infty(\mathbb{C}^n)$, we get an embedding of the bundle $H^{(k)}$ as a subbundle, say $\tilde{H}^{(k)}$ of this trivial $C^\infty(\mathbb{C}^n)$-bundle, which is preserved by $\tilde{\mathbb{D}}$. Hence we get an induced connection $\mathbb{D}$ in $H^{(k)}$. The covariant constant sections of $H^{(k)}$ with respect to $\mathbb{D}$ are identified with the Theta-functions under this embedding. From this it follows that this connection is flat. This connection can according to [Ram] and [Wel] be identified with the flat L_2-induced connection in $\tilde{H}^{(k)}$. Parallel transport in $H^{(k)}$ with respect to the connection $\mathbb{D}$ provides a canonical identification of the geometric quantization of M_I for varying I, since $\mathbb{D}$ is flat and $\mathbb{H}$ is contractible. - Since the Theta-functions are covariant constant, they explicitly realize this identification. The usual action of $Sp(2n, \mathbb{Z})$ on Theta-functions induces an action of A on the bundle $H^{(k)}$ which covers the A-action on $\mathcal{C} \cong \mathbb{H}$.

We can explicitly compute the inner products $(\Theta_{\alpha,k}, \Theta_{\beta,k})$, by first interchanging the sum and integral by absolute convergence of the sum, then doing the x-integral and finally evaluating the remaining sum of y-integrals as one Gaussian integral over $\mathbb{R}^n$:

$$(\Theta_{\alpha,k}, \Theta_{\beta,k}) = \begin{cases} 0 & \alpha \neq \beta \mod \mathbb{Z}^n \\ (2^n k^n |Y|)^{-1/2} & \text{otherwise} \end{cases}$$

where $|Y| = \det(Y)$.

Let us therefore introduce the norm $(\cdot,\cdot)_Y = (2^n k^n |Y|)^{1/2}(\cdot,\cdot)$, with respect to which $\Theta_{\alpha,k}(Z)$ give an ortho-normal basis of $H^0(M_{I(Z)}, \mathcal{L}^k_{I(Z)})$. This gives $H^{(k)}$ a Hermitian structure compatible with $\mathbb{D}$.

Let $(r,s) \in \mathbb{Z}^n \times \mathbb{Z}^n$ and consider the function $F_{r,s} \in C^\infty(M)$ given in the coordinates (x,y) by

$$F_{r,s}(x,y) = e^{2\pi i (r\cdot x + s\cdot y)}.$$

We shall now study the Toeplitz operators $T^{(k)}_{F_{r,s}}$ associated to the function $F_{r,s}$. In fact we will compute the matrix by which $T^{(k)}_{F_{r,s}}$ acts on $H^0(M_I, \mathcal{L}^k_I)$ relative to the Theta-function basis $\Theta_{\alpha,k}$. Hence we just need to compute $(F_{r,s}\Theta_{\alpha,k}, \Theta_{\beta,k})$. This is done similarly to the computation above. Using absolute convergence, the sum and the integration is interchanged. Then the x-integral gives zero unless $\alpha - \beta = -[\frac{r}{k}]$, where $[\frac{r}{k}]$ means the residue class of $\frac{r}{k}$ mod $\mathbb{Z}^n$. The remaining single sum of y-integrals rewrites to a single Gaussian integral over $\mathbb{R}^n$ just as above, and we get the result that

$$(1) \qquad (F_{r,s}\Theta_{\alpha,k}, \Theta_{\beta,k})_Y = \delta_{\alpha-\beta, -[\frac{r}{k}]} e^{-\frac{\pi i}{k} r\cdot \overline{Z} r} e^{-2\pi i s\cdot\alpha} e^{-\pi^2 (s-\overline{Z}r)\cdot(2\pi k Y)^{-1}(s-\overline{Z}r)}.$$

Hence in the basis $\Theta_{\alpha,k}(Z)$ the matrix coefficients $(T^{(k)}_{F_{r,s}})_{\alpha,\beta}$ are given by $(F_{r,s}\Theta_{\alpha,k}, \Theta_{\beta,k})_Y$. A simple rewriting gives

$$(T^{(k)}_{f(r,s,Z)(k)F_{r,s}})_{\alpha,\beta} = \delta_{\alpha-\beta,-[\frac{r}{k}]} e^{-\frac{\pi i}{k} r\cdot s} e^{-2\pi i s\cdot \alpha},$$

where

$$f(r,s,Z)(k) = e^{\frac{\pi}{2k}(s+Xr)\cdot Y^{-1}(s+Xr)} e^{\frac{\pi}{2k} r\cdot Y r}.$$

Remark 1. *The Toeplitz operators $T^{(k)}_{F_{r,s}}$ are sections of End$(H^{(k)})$ over $\mathcal{C}$. The flat connection $\mathbb{D}$ induces a flat connections $\mathbb{D}^e$ in the bundle End$(H^{(k)})$, with respect to which we see that $T^{(k)}_{F_{r,s}}$* is not *covariant constant. However the operators $T^{(k)}_{f(r,s,Z)(k)F_{r,s}}$ are covariant constant.*

Proposition 1. *Let $\Delta_{I(Z)}$ be the Laplace operator with respect to the metric*

$$g_{I(Z)}(\cdot,\cdot) = 4\pi\omega(\cdot, I(Z)\cdot)$$

on M. Then

$$e^{-\frac{1}{2k}\Delta_{I(Z)}} F_{r,s} = f(r,s,Z)(k) F_{r,s}.$$

Proof. We recall that

$$\Delta_{I(Z)} = -\frac{1}{4\pi}\left\{ (\frac{\partial}{\partial y} + X\frac{\partial}{\partial x}) \cdot Y^{-1}(\frac{\partial}{\partial y} + X\frac{\partial}{\partial x}) + \frac{\partial}{\partial x}\cdot Y\frac{\partial}{\partial x}\right\}.$$

We compute that

$$(\frac{\partial}{\partial y} + X\frac{\partial}{\partial x}) \cdot Y^{-1}(\frac{\partial}{\partial y} + X\frac{\partial}{\partial x}) F_{r,s}(x,y) = -4\pi^2\left((s+Xr)\cdot Y^{-1}(s+Xr)\right) F_{r,s}(x,y)$$

and that

$$\frac{\partial}{\partial x} \cdot Y \frac{\partial}{\partial x} F_{r,s}(x,y) = -4\pi^2 \, (r \cdot Yr) \, F_{r,s}(x,y).$$

Hence we obtain the desired formula.

□

By substituting h for k^{-1} we obtain the formal transform

$$E_I = e^{-\frac{h}{2}\Delta_I} : C^\infty_h(M) \to C^\infty_h(M).$$

We observe that E_I, $I \in \mathcal{C}$ is A-equivariant, since for all $a \in A$, we have that

$$a^* \circ E_I = E_{(a(I))} a^*.$$

Theorem 5. *Let $\star_I$, be the $*$-product obtained by applying Berezin-Toeplitz deformation quantization to M_I. Then for $f, g \in C^\infty(M)$, we have that*

$$E_I^{-1}(E_I(f) \star_I E_I(g)) = f * g,$$

where $*$ is the Moyal-Weyl product.

Proof. Define $\hat{T}^{(k)}_{F_{r,s}} = T^{(k)}_{e^{-\frac{1}{2k}\Delta_I} F_{r,s}} : H^0(M_I, \mathcal{L}^k_I) \to H^0(M_I, \mathcal{L}^k_I)$. Then

$$(\hat{T}^{(k)}_{F_{r,s}})_{\alpha,\beta} = \delta_{\alpha-\beta,-[\frac{r}{k}]} e^{-\frac{\pi i}{k} r \cdot s} e^{-2\pi i s \cdot \alpha}.$$

We then compute that

$$\begin{aligned}(\hat{T}^{(k)}_{F_{r,s}} \hat{T}^{(k)}_{F_{t,u}})_{\alpha,\beta} &= \delta_{\alpha-\beta,-[\frac{r+t}{k}]} e^{-\frac{\pi i}{k} r \cdot s} e^{-\frac{\pi i}{k} t \cdot u} e^{-\frac{2\pi i}{k} u \cdot r} e^{-2\pi i (s+u)\cdot\alpha} \\ &= e^{-\frac{\pi i}{k}(r \cdot u - t \cdot s)} (\hat{T}^{(k)}_{F_{r+t,s+u}})_{\alpha,\beta}\end{aligned}$$

Hence we see that

$$T^{(k)}_{e^{-\frac{1}{2k}\Delta_I} F_{r,s}} T^{(k)}_{e^{-\frac{1}{2k}\Delta_I} F_{t,u}} = e^{-\frac{\pi i}{k}(r \cdot u - t \cdot s)} T^{(k)}_{e^{-\frac{1}{2k}\Delta_I} F_{r+t,s+u}},$$

so

$$E_I(F_{r,s}) \star_I E_I(F_{t,u}) = e^{\pi i h (r \cdot u - t \cdot s)} E_I(F_{r+t,s+u}).$$

By Lemma 1, the result now follows.

□

Corollary 1. *We have the following relation between Toeplitz operators*

$$T^{(k)}_{E_I(F_{r,s})(1/k)} T^{(k)}_{E_I(F_{t,u})(1/k)} = T^{(k)}_{(E_I(F_{r,s}) \star_I E_I(F_{t,u}))(1/k)}. \tag{2}$$

Remark 2. *Comparing E_I with Tuynman's result, Theorem 1, we see that to first order E_I transforms the Toeplitz operators to the classical geometric quantization operators.*

Remark 3. *Combining the result of Karabegov and Schlichenmaier stated in Theorem 3 with the above, we see that $E_I \circ B_I$ provides an equivalence between the $*$-product with separation of variables $\star'_I$ on M_I, whose Karabegov form is $-\frac{1}{h}\omega$ and the Moyal-Weyl $*$-product.*

5. The topological deformation quantization of the abelian moduli space

Let Σ be a closed oriented surface of genus g. Let M be the moduli space of flat $U(1)$-connections on Σ. Then

$$M = \mathrm{Hom}(\pi_1(\Sigma), U(1)) = H^1(\Sigma, \mathbb{R})/H^1(\Sigma, \mathbb{Z}).$$

There is a symplectic structure ω on $H^1(\Sigma, \mathbb{R})$ given by the cup product, followed by evaluation on the fundamental class of Σ. By Poincare duality, we have that ω is integral and unimodular over the lattice $H^1(\Sigma, \mathbb{Z})$. The mapping class group Γ of Σ acts on M via the induced homomorphism

$$\rho : \Gamma \to \mathrm{Aut}(H^1(\Sigma, \mathbb{Z}), \omega).$$

The homomorphism ρ is surjective and has the Toreilli subgroup of Γ as its kernel.

Let p be a point on Σ and let $\Sigma' = \Sigma - \{p\}$. Since $U(1)$ is abelian we have that

$$M = \mathrm{Hom}(\pi_1(\Sigma'), U(1)).$$

In [AMR1] and [AMR2] we constructed in collaboration with Mattes and Reshetikhin a $*$-product on the Poisson manifolds

$$M^G = \mathrm{Hom}(\pi_1(\Sigma'), G)/G$$

where G is either $GL(m, \mathbb{C})$ or $SL(m, \mathbb{C})$ and the Poisson structure is determined by the choice of an invariant symmetric bilinear form on the Lie algebra of G. Notice that we in the abelian case $G = GL(1, \mathbb{C}) = \mathbb{C}^*$ have $M \subset M^G$.

We will in this section argue that this $*$-product on M^G for $G = \mathbb{C}^*$ restricts to a $*$-product on M, and we will show that the resulting $*$-product is the Moyal-Weyl product. Let us review the constructions in [AMR1] and [AMR2].

First we recall the description of the Poisson structure on M^G in terms of chord diagrams on the surface given in [AMR1].

Definition 3. *A* chord diagram *is a graph consisting of disjoint oriented circles $S_i, i \in \{1, ..., n\}$ and disjoint arcs $C_j, j \in \{1, ..., m\}$ such that:*

1. *the endpoints of the arcs are distinct*
2. $\cup_j \partial C_j = (\cup_i S_i) \cap (\cup_j C_j)$

The arcs are called chords*, the circles S_i are called the* core components *of the diagram.*

Definition 4. *A* geometrical chord diagram *on Σ' is a smooth map from a chord diagram D to Σ', mapping the chords to points. A* chord diagram on Σ' *is a class of geometric chord diagrams modulo homotopy.*

Definition 5. *By a* generic chord diagram *(on Σ') we will mean a geometrical chord diagram on Σ' such that all circles are immersed, and with all double points transverse.*

Clearly every chord diagram on Σ' contains generic chord diagrams.

Consider the complex vector space $V_{\Sigma'}$ with the basis given by the set of chord diagrams on Σ' and the subspace $W_{\Sigma'}$ generated by the 4T-relations (see [AMR1]).

Definition 6. *The algebra $ch(\Sigma') := V_{\Sigma'}/W_{\Sigma'}$ is called the* algebra of chord diagrams on Σ'.

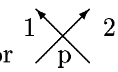
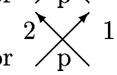
It has a natural ring structure with multiplication given by union of chord diagrams, with unit the empty diagram.

These rings are graded by the number of chords

$$ch\,(\Sigma') = \bigoplus_{n\geq 0} ch^{(n)}\,(\Sigma')$$

and we have an associated filtered space with filtered components $ch_m\,(\Sigma') := \bigoplus_{n\geq m} ch^{(n)}\,(\Sigma')$ and completion $\overline{ch\,(\Sigma')} = \prod_{n\geq 0} ch^{(n)}\,(\Sigma')$.

Recall from [AMR1] that $ch\,(\Sigma')$ has a natural Poisson structure given as follows: Assume $D_1 \cup D_2$ is a generic chord diagram. For $p \in D_1 \cap D_2$ we define the oriented intersection number by

$\epsilon_{12}(p) :=$
$+1$ for 
-1 for 

where 1 and 2 indicate components of the corresponding diagrams.

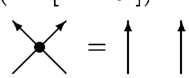
For each $p \in D_1 \cap D_2$ we define $D_1 \cup_p D_2$ to be the chord diagram on Σ' given by joining $D_1^{-1}(p)$ and $D_2^{-1}(p)$ by a chord. Under the above assumptions we define their Poisson bracket to be

$$\{[D_1], [D_2]\} := \sum_{p\in D_1\cap D_2} \epsilon_{12}(p)[D_1 \cup_p D_2] \tag{3}$$

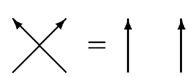
It is closely related to the Poisson structure on the moduli space of flat G-connections on Σ', where G is a complex Lie group with an invariant bilinear pairing on its Lie algebra. The following is one of the main results of [AMR1]:

Theorem 6. *Given a finite dimensional representation of G, there is a Poisson algebra homomorphism F (given by formula (2) in* [AMR1]*) from $ch\,(\Sigma')$ to the Poisson algebra $\mathcal{O}(M^G)$ of algebraic functions on the moduli space of flat G-connections on Σ'. This homomorphism is in many interesting cases surjective, including the case of G being $GL(m,\mathbb{C})$ or $SL(m,\mathbb{C})$ and the representation being the defining representation.*

A connected geometric chord diagram D with zero chords is just a closed curve on Σ', and the function F_D associated to this diagram is simply just the holonomy function of the curve D. For non-connected diagrams with zero chords, the function is simply just the product of the holonomy functions for each component. Since we have the following local relation (see [AMR2]) for $G = GL(m,\mathbb{C})$

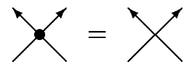
(4) 

we see that this determines the homomorphism F completely in this case. In the very special case of $G = GL(1,\mathbb{C})$, which interest us here the relation, which we get from the Cayley-Hamilton theorem, is the following very simple relation (see [AMR2])

(5) 

This tells us that for a chord diagram D on Σ', the function F_D only depends on the integer homology classes of $[D] \in H^1(\Sigma,\mathbb{Z})$, that is the sum in integer homology of the images of the core components. For $\gamma \in H^1(\Sigma, \mathbb{Z})$ we simply use the notation F_γ for F_D for any chord diagram D such that $[D] = \gamma$.

Further, if we combine the two relations we get of course

(6) 

which we will see gives an enormous simplification, when we compute the $*$-product.

Let us now recall from [AMR2], how we quantize the Poisson algebra of chord diagrams and how this induces a $*$-product on M^G.

Denote by $L(\Sigma')$ the $\mathbb{C}$-vector space spanned by (framed) links in $\Sigma' \times [0,1]$ (by a link we mean an isotopy class of smooth imbeddings $(S^1)^{\cup} \hookrightarrow \Sigma' \times [0,1]$) We have a multiplication on $L(\Sigma')$ defined by the following rule: Let L_1 and L_2 be links in $\Sigma' \times [0,1]$. Isotope L_1 such that it is contained in $\Sigma' \times [0, \frac{1}{2}]$ and L_2 such that it is contained in $\Sigma' \times [\frac{1}{2}, 1]$. Then define

$$L_1.L_2 = L_1 \cup L_2$$

where $L_1 \cup L_2 \subset (\Sigma' \times [0,1/2]) \cup_{\Sigma' \times \{1/2\}} (\Sigma' \times [1/2,1]) = \Sigma' \times [0,1]$.

This multiplication determines on $L(\Sigma')$ the structure of an associative (in general noncommutative) ring with the empty link being the unit element.

Let us now recall the Vassiliev filtration on $L(\Sigma')$. Let $L \subset \Sigma' \times [0,1]$ be a link and $D_L \subset \Sigma'$ some link diagram of L, so that D_L is (an isotopy class of) a regular projection of L to Σ'. As usual we distinguish vertices of two types: For each vertex $v \in D_L$ we introduce an oriented crossing number $\epsilon(v)$ such that a positive crossing gives $+1$ and a negative gives -1.

If the diagram D_L has vertices $v_1, \dots, v_n$ with corresponding oriented crossing numbers $\epsilon_1, \dots, \epsilon_n$ we also denote it by $D_L^{\epsilon_1,\dots,\epsilon_n}$ when we wish to emphasize the types of the crossings. We may regard L as an equivalence class $[D_L]$ of diagrams that are related by Reidemeister moves.

Introduce the following operation ∇: Choose a set of crossings $v_{i_1}, \dots, v_{i_m}$ of D_L and set

$$\nabla_{v_{i_1},\dots,v_{i_m}} D_L := \sum_{\epsilon_{i_1},\dots,\epsilon_{i_m} = \pm 1} \epsilon_{i_1} \dots \epsilon_{i_m} [D_L^{\epsilon_1,\dots,\epsilon_n}]$$

This maps the link diagram D_L to a linear combination of links whose regular projections are obtained from D_L by switching crossings.

Now let $L_m(\Sigma') \subseteq L(\Sigma')$ be the span of all elements of the form $\nabla_{v_{i_1},\dots,v_{i_m}} D_L$ where D_L runs over all possible link diagrams.

The filtration $L(\Sigma') \supset L_1(\Sigma') \supset L_2(\Sigma') \dots$ is compatible with the algebra structure. We get a Poisson structure on

$$L_{Gr}(\Sigma') = \bigoplus_{n \geq 0} L_n(\Sigma')/L_{n+1}(\Sigma'),$$

since if $x \in L_n$ and $x' \in L_m$ then $x.x' - x'.x \in L_{m+n+1}$ and we define

$$\{[x],[x']\} = [x.x' - x'.x] \in L_{m+n+1}(\Sigma')/L_{m+n+2}(\Sigma').$$

Set $L_\infty(\Sigma') := \bigcap_{n \in \mathbb{N}} L_n(\Sigma')$ and $L'(\Sigma') := L(\Sigma')/L_\infty(\Sigma')$.

To any element $D \in ch^{(n)}(\Sigma')$ we can associate an element $\lambda(D) \in L(\Sigma')/L_{n+1}(\Sigma')$ by setting

$$\lambda(D) := \nabla_{v_{i_1},\dots,v_{i_n}} D_L \text{ mod } L_{n+1}(\Sigma')$$

for any link L that projects to the diagram D, where $\{v_{i_1}, \dots, v_{i_n}\}$ is the set of chords of D. This defines a graded linear map $\lambda : ch(\Sigma') \to L_{Gr}(\Sigma')$, which according to Proposition 9 in [AMR2] is a graded Poisson homomorphism and in fact an isomorphism, since there exists a universal Vassiliev invariant for Σ':

A universal Vassiliev invariant is a filtration preserving linear map $V : L(\Sigma') \to \overline{ch(\Sigma')}$ such that

$$V(\lambda(D)) = D \bmod \overline{ch_{k+1}(\Sigma')},$$

for all $D \in ch^{(k)}(\Sigma')$. By Theorem 12 in [AMR2] we have that such an invariant exists.

The $*$-product on chord diagrams is now induced via a universal Vassiliev invariant from the product on $L(\Sigma')$ as follows:

The extension $\overline{V} : \overline{L'(\Sigma)} \to \overline{ch(\Sigma')}$ is by Theorem 22 in [AMR2] an isomorphism and according to Theorem 24 and formula (7) in [AMR2] we define the $*$-product on $ch(\Sigma)[[h]]$ to be

$$D_1 * D_2 = h^{-deg(D_1 D_2)} \sum_{i=1}^{\infty} \overline{V}(\overline{V}^{-1}(D_1).\overline{V}^{-1}(D_2))^{(i)} h^i,$$

where $\overline{V}(L) = \sum_{i=0}^{\infty} \overline{V}(L)^{(i)}$ and $\overline{V}(L)^{(i)} \in ch^{(i)}(\Sigma)$.

The $*$ product on $\mathcal{O}(M^G)[[h]]$ is defined as follows. Let D_1 and D_2 be two chord diagrams on Σ'. Then $F_{D_1}, F_{D_2} \in \mathcal{O}(M^G)$ and we define

$$F_{D_1} *_{\text{Top}} F_{D_2} = F_{D_1 * D_2}.$$

By theorem 29 in [AMR2] this is well-defined and by theorem 10 in [AMR1], this determines the $*$-product on M^G uniquely. For $G = \mathbb{C}^*$ we have the following result.

Theorem 7. *For any two elements λ and λ' in $H_1(\Sigma, \mathbb{Z})$, we have that*

$$F_\lambda *_{Top} F_{\lambda'} = \exp(\pi i h \omega(\lambda, \lambda')) F_{\lambda + \lambda'},$$

hence $*_{Top}$ restricts to the Moyal-Weyl $*$-product on $M \subset M^G$, $G = \mathbb{C}^*$.

Proof. We refer the reader to section 3.3 in [AMR2] for the construction of the universal Vassiliev invariant for Σ'. The important point is however now that the relation (6) means that a chord only contributes a factor h. This has the effect, that the associator is mapped to the identity morphism. This is easily seen from the formula for the associator in [LM]. Furthermore the $\pm$-crossing morphisms are simply just mapped to $\exp(\pm\frac{h}{2})$ times the identity morphism.

Suppose now that D and D' are geometric chord diagrams in generic position on the surface Σ' which represents γ and γ' respectively. When we compute $\overline{V}(\overline{V}^{-1}(D).\overline{V}^{-1}(D'))^{(i)}$ we can ignore the associators. The only contribution we are left with is therefore a factor of $\exp(\pm\frac{h}{2})$ times $\gamma + \gamma'$ for each intersection p between D and D', the sign being equal to the sign of the intersection at p between D and D'. The theorem follows directly from this.

□

6. Relations to Abelian Chern-Simons theory

In $2+1$-dimensional Chern-Simons theory, the 2-dimensional part of the theory is a modular functor, which is a functor from the category of compact smooth oriented surfaces to the category of finite dimensional complex vector spaces, which satisfies certain properties. In the gauge-theoretic construction of this functor one first fixes a compact Lie group K and an invariant non-degenerate inner product on its Lie algebra. The functor then associates to a closed oriented surface the finite dimensional vector space one obtains by applying geometric quantization to the moduli space of flat K-connections on the surface (See e.g. [W1] and [At1]). - In the abelian case $K = U(1)$ at hand this means concretely the following. For a closed oriented surface Σ we recall from the previous section that

$$M = H^1(\Sigma, \mathbb{R})/H^1(\Sigma, \mathbb{Z})$$

and the symplectic structure on M is introduced from that on $H^1(\Sigma, \mathbb{R})$, which is integral on $H^1(\Sigma, \mathbb{Z})$ and unimodular. Hence we can apply the discussion in section 4 to our abelian moduli space M. Thus we have the Hermitian vector bundle $H^{(k)}$ over the space of complex structures $\mathcal{C}$ on $H^1(\Sigma, \mathbb{R})$. This bundle has a flat connection, and an action of $\mathrm{Aut}(H^1(\Sigma, \mathbb{Z}), \omega)$, which preserves the Hermitian structure and the flat connection. In this case the modular functor is defined by associating to Σ, the vector space $Z_k(\Sigma)$ consisting of covariant constant sections of $H^{(k)}$ over $\mathcal{C}$. So through the representation ρ, we get a representation ρ_k of the mapping class group Γ of Σ on $Z_k(\Sigma)$.

In $2+1$-dimensional Chern-Simons theory one also has the following TQFT setup. Suppose Y is a compact oriented 3-manifold such that $\partial Y = (-\Sigma_1) \cup \Sigma_2$, where Σ_1 and Σ_2 are closed oriented surfaces and $-\Sigma_1$ means Σ_1 with the orientation reversed. Assume further L is a link inside $Y \setminus \partial Y$. Then the TQFT-axioms states that there should be a linear morphism $Z_k(Y, L) : Z_k(\Sigma_1) \to Z_k(\Sigma_2)$, which satisfies that glueing along boundary components goes to the corresponding composition of linear maps. We will give a geometric construction of these operators in the case where $Y = \Sigma \times [0,1]$.

Let us first provide a geometric construction of the morphisms

$$Z_k(\gamma) = Z_k(\Sigma \times [0,1], \gamma) \in \mathrm{Hom}(Z_k(\Sigma)),$$

for any simple closed curve γ on the surface $\Sigma \times \{1/2\} \cong \Sigma$.

To a simple closed curve γ on Σ, we consider the holonomy function $F_\gamma \in C^\infty(M)$. This is a pure phase function, hence the series $E_I(F_\gamma) \in C_h^\infty(M)$ is convergent for $h = 1/k$ for all $I \in \mathcal{C}$. According to Remark 1 and Proposition 1, the section of $\mathrm{End}(H^{(k)})$ given by $T^{(k)}_{E_I(F_\gamma)(1/k)}$ at $I \in \mathcal{C}$ is covariant constant. We define $Z_k(\gamma)$ to be this section.

Let us now use these operators to give a TQFT proof of the following well known classical result from the theory of Theta-functions.

Theorem 8. *We have that*

$$\bigcap_{k=1}^{\infty} \ker \rho_k = \ker \rho$$

is the Toreilli subgroup of Γ.

Proof. Suppose we have a $\phi \in \Gamma$. Then ϕ induces the sympletomorphism $\rho(\phi)$ of M. For any simple closed curve on Σ, we get the following commutative diagram

$$\begin{array}{ccc} Z_k(\Sigma) & \xrightarrow{\rho_k(\phi)} & Z_k(\Sigma) \\ {\scriptstyle Z_k(\gamma)}\big\downarrow & & {\scriptstyle Z_k(\phi(\gamma))}\big\downarrow \\ Z_k(\Sigma) & \xrightarrow{\rho_k(\phi)} & Z_k(\Sigma). \end{array}$$

Suppose now $\phi \in \bigcap_{k=1}^{\infty} \ker \rho_k$, then $Z_k(\gamma) = Z_k(\phi(\gamma))$. But this means that

$$T^{(k)}_{E_I(F_\gamma)(1/k)} = T^{(k)}_{E_I(F_{\phi(\gamma)})(1/k)}$$

for all k and all $I \in \mathcal{C}$. Hence we get that

$$\lim_{k\to\infty} \|T^{(k)}_{F_\gamma - F_{\phi(\gamma)}}\| = 0.$$

By Bordemann, Meinrenken and Schlichenmaier's theorem 4, we must have that $F_\gamma = F_{\phi(\gamma)}$. But then since $F_{\phi(\gamma)} = F_\gamma \circ \rho(\phi)$, we see that $\rho(\phi)$ acts trivial on $C^\infty(M)$. Then we must have that $\phi \in \ker \rho$.

□

Remark 4. *In* [A1] *we have generalized a part of the constructions presented here to the non-abelian case of $K = SU(n)$ and proved an analog of theorem 8, which in that case yields the much stronger asymptotic faithfulness result, that the intersection of the kernels is trivial. See Theorem 1 in* [A1] *for the precise statement.*

Let us now return to the TQFT constructions and give a geometric construction of

$$Z_k(\Sigma \times [0,1], L) \in \mathrm{Hom}(Z_k(\Sigma))$$

for any link L as follows. Consider $\overline{V}(L)$ as an element in $H^1(\Sigma, \mathbb{Z})[[h]]$. By the arguments in the proof of Theorem 7, we see that we only get contributions from the crossings, and since these contribute with convergent power series in h, we see that $F_L := F_{\overline{V}(L)} \in C^\infty(M)[[h]]$, is actually convergent for any h. But then we can simply define $Z_k(\Sigma \times [0,1], L) \in \mathrm{Hom}(Z_k(\Sigma))$ to be the covariant constant section $T^{(k)}_{E_I(F_L)(1/k)}$ of $\mathrm{End}(H^{(k)})$ over $\mathcal{C}$. Now all of the above results combine to give the following Theorem

Theorem 9. *We have that the operators $Z_k(\Sigma \times [0,1], L) \in Hom(Z_k(\Sigma))$ satisfy the glueing law*

$$Z_k(\Sigma \times [0,1], L_1.L_2) = Z_k(\Sigma \times [0,1], L_1) \circ Z_k(\Sigma \times [0,1], L_2).$$

for any two link L_1 and L_2 in $\Sigma \times [0,1]$.

Proof. By the very definition of the $*$-product $*_{\mathrm{Top}}$ on the moduli space M, we have for any two links L_1 and L_2, that

$$F_{L_1.L_2} = F_{L_1} *_{\mathrm{Top}} F_{L_2}.$$

But then by Theorem 7 and Corollary 1 we get that

$$T^{(k)}_{E_I(F_{L_1.L_2})(1/k)} = T^{(k)}_{E_I(F_{L_1})(1/k)} \circ T^{(k)}_{E_I(F_{L_2})(1/k)},$$

for all $I \in \mathcal{C}$. The glueing law follows from this.

□

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