THE CLASS OF DISTRIBUTIONS OF PERIODIC ORNSTEIN-UHLENBECK PROCESSES DRIVEN BY LÉVY PROCESSES

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ABSTRACT. The class $I(c)$ of stationary distributions of periodic Ornstein-Uhlenbeck processes with parameter c driven by Lévy processes is analyzed. A characterization of $I(c)$ analogous to a well-known characterization of the selfdecomposable distributions is given. The relations between $I(c)$ for varying values of c and the relations with the class of selfdecomposable distributions and with the nested classes L_m are discussed.

1. INTRODUCTION

Let $\{Z_t\}_{t\in[0,1]}$ be a Lévy process restricted to $t \in [0,1]$ with values in $\mathbb{R}^d$. Let $c \in \mathbb{R} \setminus \{0\}$. Consider the Langevin equation

$$\mathrm{d}X_t = -cX_t\mathrm{d}t + \mathrm{d}Z_t, \qquad t \in [0,1] \tag{1.1}$$

with boundary condition

$$X_0 = X_1 \qquad \text{a.s.} \tag{1.2}$$

This has a unique solution

$$X_t = e^{-ct}X_0 + e^{-ct}\int_0^t e^{cs}\mathrm{d}Z_s, \qquad t \in [0,1] \tag{1.3}$$

with

$$X_0 = X_1 = \frac{1}{e^c - 1}\int_0^1 e^{cs}\mathrm{d}Z_s. \tag{1.4}$$

All equalities involving stochastic integrals are understood in the sense "almost surely". We call $\{X_t\}_{t\in[0,1]}$ the *periodic Ornstein-Uhlenbeck process* with parameter c and background driving Lévy process $\{Z_t\}_{t\in[0,1]}$. The pathwise periodic extension $\{\widetilde{X}_t\}_{t\in\mathbb{R}}$ of $\{X_t\}_{t\in[0,1]}$ by $\widetilde{X}_t = X_t$ for $t \in [0,1]$ and $\widetilde{X}_{t+n} = \widetilde{X}_t$ for $t \in \mathbb{R}$ and integers n is

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a stationary process, that is, $\{\widetilde{X}_t\}_{t\in\mathbb{R}}$ and $\{\widetilde{X}_{t+s}\}_{t\in\mathbb{R}}$ have an identical system of finite-dimensional distributions for any $s \in \mathbb{R}$. In particular, $\mathcal{L}(X_t) = \mathcal{L}(X_1) = \mathcal{L}\left(\frac{1}{e^c-1}\int_0^1 e^{cs}\mathrm{d}Z_s\right)$ for any $t \in [0,1]$, where we denote the distribution of a random vector Y by $\mathcal{L}(Y)$. We call this distribution the stationary distribution of $\{X_t\}_{t\in[0,1]}$. These are processes introduced and studied by Pedersen [8]. The facts above were shown in [8] in the case where $d = 1$ and $c > 0$, and proofs in the general case can be given in the same way (see also [2] and [10] for the equivalence of (1.1) and (1.3)). The Gaussian periodic Ornstein-Uhlenbeck processes appear in Kwakernaak [6], and Grenander [3] and others use these processes in relation to statistical shape analysis; see [8].

In this paper we are interested in the analysis of the class $I(c)$ of stationary distributions of periodic Ornstein-Uhlenbeck processes with parameter c where the background driving Lévy processes $\{Z_t\}_{t\in[0,1]}$ are ranging over all Lévy processes on $\mathbb{R}^d$. This continues the study initiated by Pedersen [8]. A usual Ornstein-Uhlenbeck process with parameter $c > 0$ and background driving Lévy process $\{Z_t\}_{t\in[0,\infty)}$ is a solution of equation (1.1) with $t \in [0,1]$ replaced by $t \in [0,\infty)$ under the condition that X_0 is independent of the process $\{Z_t\}_{t\in[0,\infty)}$. It is determined uniquely by X_0 and expressed by (1.3) with $t \in [0,1]$ replaced by $t \in [0,\infty)$. It satisfies $\mathcal{L}(X_t) = \mathcal{L}(X_0)$ for $t \in [0,\infty)$ if and only if $\mathcal{L}(Z_1)$ has finite log-moment and $\mathcal{L}(X_0) = \mathcal{L}\left(\int_0^\infty e^{-cs}\mathrm{d}Z_s\right)$. In this case the stationary distribution $\mu = \mathcal{L}(X_0)$ is selfdecomposable. Conversely, any selfdecomposable distribution μ is expressed in this way with a Lévy process $\{Z_t\}_{t\in[0,\infty)}$ having finite log-moment. See [4] or [12]. The correspondence between μ and $\mathcal{L}(Z_1)$ is one-to-one for fixed $c > 0$. Thus the class of stationary distributions of usual Ornstein-Uhlenbeck processes does not depend on the parameter c and coincides with the class $SD(\mathbb{R}^d)$ of selfdecomposable distributions on $\mathbb{R}^d$. However, we will see that the class $I(c)$ delicately depends on the parameter c.

In the next section we will characterize $I(c)$ in terms of properties of the Lévy measures of the distributions in this class. The mapping Φ_c of $\mu_0 = \mathcal{L}(Z_1)$ to $\mu = \Phi_c(\mu_0) = \mathcal{L}\left(\int_0^1 e^{cs}\mathrm{d}Z_s\right)$ is shown to be one-to-one and a homeomorphism. In Section 3 we will study the relations between $I(c)$ for varying values of c and the relations between $I(c)$ and $SD(\mathbb{R}^d)$. It will be shown that $I(c) \supsetneqq SD(\mathbb{R}^d)$ and that $\bigcap_{n=1}^\infty I(c_n) = SD(\mathbb{R}^d)$ whenever $c_n \uparrow \infty$. One way of looking at the dependence of $I(c)$ on c is to study, given μ in the class $ID(\mathbb{R}^d)$ of infinitely divisible distributions on $\mathbb{R}^d$, the set $Q(\mu) = \{c \geqslant 0\colon \mu \in I(c)\}$, where we define $I(0) = ID(\mathbb{R}^d)$. We will show that these sets $Q(\mu)$ have a rich variety. In Section 4 we consider the relations to other classes of distributions. The decreasing sequence of classes L_m, $m = 0, 1, \ldots, \infty$, beginning with $L_0 = SD(\mathbb{R}^d)$ is considered. These classes are known to be intimately connected with the mapping $\widetilde{\Phi}_{-c}$ of $\mu_0 = \mathcal{L}(Z_1)$ to $\mu = \widetilde{\Phi}_{-c}(\mu_0) = \mathcal{L}\left(\int_0^\infty e^{-ct}\mathrm{d}Z_t\right)$, but the relations with Φ_c are more subtle; we will show that $L_m \subsetneqq \Phi_c(L_{m-1}) \subsetneqq L_{m-1}$ for any finite m.

Stochastic integrals $\int_0^t f(s)\mathrm{d}Z_s$ in this paper are defined as limits in probability from the case of step functions f, as in [7], [9], [10], [13], [17]. The integral $\int_0^\infty f(s)\mathrm{d}Z_s$ is defined to be the limit in probability of $\int_0^t f(s)\mathrm{d}Z_s$ as $t \to \infty$ whenever it exists.

2. Characterization and properties of class $I(c)$

Throughout the paper fix d in the set $\mathbb{N}$ of positive integers. Elements of $\mathbb{R}^d$ are column vectors. The inner product on $\mathbb{R}^d$ is denoted $\langle x, y\rangle$ and the corresponding norm is $|x|$. For a distribution μ on $\mathbb{R}^d$ denote the characteristic function of μ by $\widehat{\mu}$, $\widehat{\mu}(z) = \int_{\mathbb{R}^d} e^{\mathrm{i}\langle z,x\rangle}\mu(\mathrm{d}x)$, $z \in \mathbb{R}^d$. For probability measures μ_n $(n = 1, 2, \ldots)$ and μ on $\mathbb{R}^d$, $\mu_n \to \mu$ means weak convergence of μ_n to μ. Let $D = \{x \in \mathbb{R}^d \colon |x| \leqslant 1\}$. For $\mu_0 \in ID(\mathbb{R}^d)$ let $\Psi_0(z) = \log\widehat{\mu}_0(z)$, the distinguished logarithm of $\widehat{\mu}_0$ in [12], p. 33. We have

$$\Psi_0(z) = -\frac{1}{2}\langle z, A_0 z\rangle + \mathrm{i}\langle\gamma_0, z\rangle + \int_{\mathbb{R}^d}\left(e^{\mathrm{i}\langle z,x\rangle} - 1 - \mathrm{i}\langle z, x\rangle 1_D(x)\right)\nu_0(\mathrm{d}x), \quad z \in \mathbb{R}^d,$$

where (A_0, ν_0, γ_0) is the triplet of μ_0. For $s \geqslant 0$ let $\mu_0^s = \mu^{*s}$, which has characteristic function $\widehat{\mu}_0^s(z) = \exp s\Psi_0(z)$, $z \in \mathbb{R}^d$.

If $f\colon \mathbb{R}_+ \longrightarrow \mathbb{R}$ is bounded and measurable and $\{Z_s\}_{s\geqslant 0}$ is a Lévy process on $\mathbb{R}^d$, then the stochastic integral $\int_a^b f(s)\mathrm{d}Z_s$ is definable for $0 \leqslant a \leqslant b < \infty$ and has an infinitely divisible distribution on $\mathbb{R}^d$ with

$$E\exp\left(\mathrm{i}\left\langle z, \int_a^b f(s)\mathrm{d}Z_s\right\rangle\right) = \exp\int_a^b \Psi_0(f(s)z)\mathrm{d}s, \quad z \in \mathbb{R}^d. \tag{2.1}$$

Now and then we integrate with respect to a time-changed Lévy process. Hence, for $t > 0$ and $s_0 \geqslant 0$, $\int_a^b f(s)\mathrm{d}_s Z_{s_0+ts}$ denotes the stochastic integral of f with respect to the Lévy process $\{Z_{s_0+ts} - Z_{s_0}\}_{s\geqslant 0}$.

For $m = 1, 2, \ldots$ let $ID_{\log^m}(\mathbb{R}^d)$ be the class of distributions $\mu \in ID(\mathbb{R}^d)$ for which the Lévy measure ν of μ satisfies

$$\int_{|x|>2}(\log|x|)^m\,\nu(\mathrm{d}x) < \infty. \tag{2.2}$$

Equivalently, $ID_{\log^m}(\mathbb{R}^d)$ is the class of $\mu \in ID(\mathbb{R}^d)$ such that $\int_{|x|>2} (\log |x|)^m \, \mu(\mathrm{d}x) < \infty$. Let $ID_{\log}(\mathbb{R}^d) = ID_{\log^1}(\mathbb{R}^d)$. The stochastic integral $\int_0^\infty e^{-cs}\mathrm{d}Z_s$ with $c > 0$ exists if and only if $\mathcal{L}(Z_1) \in ID_{\log}(\mathbb{R}^d)$. There is a one-to-one correspondence between $SD(\mathbb{R}^d)$ and $ID_{\log}(\mathbb{R}^d)$ as is mentioned in the previous section.

Let $\mu \in ID(\mathbb{R}^d)$ with triplet (A, ν, γ). Recall that if $\nu = 0$ then μ is said to be Gaussian, and if $A = 0$ then μ is said to be purely non-Gaussian. Let M be a class of distributions on $\mathbb{R}^d$. Following [11] we say that M is completely closed if the following three conditions (C1)–(C3) are satisfied, where (C1) $\mu_1, \mu_2 \in M$ implies $\mu_1 * \mu_2 \in M$; (C2) $\mu_n \in M$ for $n = 1, 2, \ldots$ and $\mu_n \to \mu$ imply $\mu \in M$; (C3) $\mathcal{L}(X) \in M$ implies $\mathcal{L}(aX + b) \in M$ for all $a > 0$ and $b \in \mathbb{R}^d$. Sometimes we also need the conditions (D), (P) and (GJ) on M. These are: (D) (dual property) if $\mathcal{L}(X) \in M$ then $\mathcal{L}(-X) \in M$; (P) (raising to the power) if $\mu \in M$ then $\mu^t \in M$ for all $t \geqslant 0$; (GJ) if $\mu \in M$ and $\mu = \mu_G * \mu_J$ where μ_G is Gaussian and μ_J is purely non-Gaussian, then $\mu_G, \mu_J \in M$. When we consider (P) or (GJ), we assume $M \subseteq ID(\mathbb{R}^d)$.

Definition 2.1. For $c \in \mathbb{R} \setminus \{0\}$ let $I(c)$ be the class of distributions given by

$$I(c) = \left\{ \mathcal{L}\left(\int_0^1 e^{cs}\mathrm{d}Z_s \right) : \{Z_s\}_{s \geqslant 0} \text{ is a Lévy process on } \mathbb{R}^d \right\}. \tag{2.3}$$

We have $I(c) \subseteq ID(\mathbb{R}^d)$ as is mentioned above. We define the mapping Φ_c from $ID(\mathbb{R}^d)$ onto $I(c)$ by

$$\Phi_c(\mu_0) = \mathcal{L}\left(\int_0^1 e^{cs}\mathrm{d}Z_s \right), \qquad \mu_0 = \mathcal{L}(Z_1). \tag{2.4}$$

Note that $\mu = \mathcal{L}\left(\int_0^1 e^{cs}\mathrm{d}Z_s \right) \in I(c)$ if and only if μ is the stationary distribution of a periodic Ornstein-Uhlenbeck process with parameter c and background driving Lévy process $\{Z'_s\}_{s \in [0,1]}$ where $Z'_s = (e^c - 1)Z_s$. See (1.4).

Remark 2.2. Let $t > 0$ and $c \in \mathbb{R} \setminus \{0\}$. Then,

$$I(ct) = \left\{ \mathcal{L}\left(\int_0^t e^{cs}\mathrm{d}Z_s \right) : \{Z_s\}_{s \geqslant 0} \text{ is a Lévy process on } \mathbb{R}^d \right\}.$$

Indeed, $\int_0^t e^{cs}\mathrm{d}Z_s = \int_0^1 e^{cts}\mathrm{d}_s Z_{st}$. As a special case,

$$I(c) = \left\{ \mathcal{L}\left(\int_0^c e^{s}\mathrm{d}Z_s \right) : \{Z_s\}_{s \geqslant 0} \text{ is a Lévy process on } \mathbb{R}^d \right\}$$

for $c > 0$. Thus $I(c)$, $c > 0$, can be understood also as the class of stationary distributions of c-periodic Ornstein-Uhlenbeck processes with parameter 1 and background driving Lévy process $\{Z'_s\}_{s\in[0,c]}$, namely, the distributions of the solutions of

$$\mathrm{d}X_t = -X_t\mathrm{d}t + \mathrm{d}Z'_t, \quad t \in [0,c], \quad \text{with } X_0 = X_c \text{ a.s.}$$

Proposition 2.3. *Let $c \in \mathbb{R}\setminus\{0\}$. Let $\mu_0 \in ID(\mathbb{R}^d)$ have triplet (A_0,ν_0,γ_0) and corresponding distinguished logarithm $\Psi_0(z)$. Let $\mu = \Phi_c(\mu_0)$. Then,*

$$\widehat{\mu}(z) = \exp\int_0^1 \Psi_0(e^{cs}z)\mathrm{d}s, \quad z \in \mathbb{R}^d, \tag{2.5}$$

and the triplet of μ is (A,ν,γ), where

$$A = \int_0^1 e^{2cs}\mathrm{d}sA_0 = \frac{e^{2c}-1}{2c}A_0, \tag{2.6}$$

$$\nu(B) = \int_{\mathbb{R}^d}\nu_0(\mathrm{d}x)\int_0^1 1_B(e^{cs}x)\mathrm{d}s, \quad B \in \mathcal{B}(\mathbb{R}^d), \tag{2.7}$$

$$\gamma = \frac{e^c-1}{c}\gamma_0 - \int_{\mathbb{R}^d}\nu_0(\mathrm{d}x)\int_0^1 e^{cs}x[1_D(x) - 1_D(e^{cs}x)]\mathrm{d}s. \tag{2.8}$$

Proof. The expression for $\widehat{\mu}(z)$ follows from (2.1). The expression for the triplet is derived in [10], p. 35, or [13]. □

The following are basic properties of $I(c)$.

Proposition 2.4. *Let $c \in \mathbb{R}\setminus\{0\}$.*

(i) *If $\mu \in I(c)$, then $\mu^t \in I(c)$ for all $t \geqslant 0$.*
(ii) *For $\mu_{0,1},\mu_{0,2} \in ID(\mathbb{R}^d)$ we have $\Phi_c(\mu_{0,1}) * \Phi_c(\mu_{0,2}) = \Phi_c(\mu_{0,1} * \mu_{0,2})$.*
(iii) *The class $I(c)$ satisfies* (C1), (C3), (D), (P) *and* (GJ).
(iv) *$I(c) = I(-c)$. In fact, if $\mu_1 = \Phi_c(\mu_0)$ and $\mu_2 = \Phi_{-c}(\mu_0)$, then $\widehat{\mu}_2(z) = \widehat{\mu}_1(e^{-c}z)$, $z \in \mathbb{R}^d$.*

Proof. (i) Let $\mu = \mathcal{L}(\int_0^1 e^{cs}\mathrm{d}Z_s)$ and $\mu_0 = \mathcal{L}(Z_1)$. Let $\Psi_0 = \log\widehat{\mu}_0$. Then $\widehat{\mu}^t(z) = \exp\left(t\int_0^1 \Psi_0(e^{cs}z)\mathrm{d}s\right)$ by Proposition 2.3. It follows that $\mu^t = \mathcal{L}\left(\int_0^1 e^{cs}\mathrm{d}_sZ_{ts}\right) \in I(c)$. The proof of (ii) is left to the reader. By (i)–(ii) $I(c)$ satisfies (C1) and (P). It is, moreover, readily seen that $I(c)$ satisfies (C3), (D) and (GJ). The proof of (iv) is given by the use of (2.5). □

Remark 2.5. As a special case of Proposition 2.10 (ii) below it follows that $I(c)$ satisfies (C2). Hence $I(c)$ is completely closed.

In the following we derive a characterization of $I(c)$. Let $S = \{\xi \in \mathbb{R}^d\colon |\xi| = 1\}$ and Leb denote the Lebesgue measure on $(0,\infty)$.

Proposition 2.6 (Polar decomposition of a Lévy measure). *Let ν be a Lévy measure on $\mathbb{R}^d$. Then, ν is decomposed in polar form as*

$$\nu(B) = \int_S \lambda(\mathrm{d}\xi) \int_0^\infty 1_B(u\xi)\nu_\xi(\mathrm{d}u), \quad B \in \mathcal{B}(\mathbb{R}^d),$$

where λ is a finite measure on S and ν_ξ is a measure on $(0,\infty)$ such that $\nu_\xi(B)$ is measurable in ξ for each $B \in \mathcal{B}((0,\infty))$ and $\int_0^\infty (1 \wedge u^2)\nu_\xi(\mathrm{d}u) = 1$ for $\xi \in \mathbb{R}^d$. This λ is unique and ν_ξ is unique for λ-a.e. ξ.

Let $c > 0$. If $\nu_\xi(\mathrm{d}u)$ has density $k_\xi(u)/u$ with respect to the Lebesgue measure on $(0,\infty)$ with some nonnegative (ξ, u)-measurable $k_\xi(u)$, then $\sum_{j=1}^\infty k_\xi(e^{jc}u) < \infty$ for $\lambda \times \mathrm{Leb}$-a.e. (ξ, u).

Proof. The polar decomposition is constructed as in the proof of Theorem 15.10 in [12]. We leave the details to the reader.

Assume $\nu_\xi(\mathrm{d}u)$ has density $k_\xi(u)/u$. Denote $G_\xi^*(u) = \sum_{j=1}^\infty k_\xi(e^{jc}u)$. We have to show that $G_\xi^*(u)$ is finite for $\lambda \times$ Leb-a. e. (ξ, u). For $a > 0$ we have

$$\begin{aligned}
&\int_S \lambda(\mathrm{d}\xi) \int_a^{e^c a} \frac{G_\xi^*(u)}{u}\mathrm{d}u = \sum_{j=1}^\infty \int_S \lambda(\mathrm{d}\xi) \int_a^{e^c a} \frac{k_\xi(e^{jc}u)}{u}\mathrm{d}u \\
&= \sum_{j=1}^\infty \int_S \lambda(\mathrm{d}\xi) \int_{e^{jc}a}^{e^{(j+1)c}a} \frac{k_\xi(u)}{u}\mathrm{d}u = \int_S \lambda(\mathrm{d}\xi) \int_{e^c a}^\infty \frac{k_\xi(u)}{u}\mathrm{d}u = \nu(\{x\colon |x| > e^{ca}\}) < \infty,
\end{aligned}$$

which gives the desired result. □

Theorem 2.7. *Let $c > 0$. Let $\mu \in ID(\mathbb{R}^d)$ have Lévy measure ν with polar decomposition $\lambda(\mathrm{d}\xi)\nu_\xi(\mathrm{d}u)$.*

(i) *μ is in $I(c)$ if and only if the following two conditions* (a)–(b) *are satisfied.*
   (a) *For λ-a.e. ξ the measure $\nu_\xi(\mathrm{d}u)$ has density $k_\xi(u)/u$ with respect to the Lebesgue measure on $(0,\infty)$ with some nonnegative (ξ, u)-measurable $k_\xi(u)$;*
   (b) *Let $G_\xi^*(u) = \sum_{j=1}^\infty k_\xi(e^{jc}u)$. Then $G_\xi^*(u)$ has a version which is decreasing in u.*

(ii) *Assume $\mu \in I(c)$. Then there is one and only one $\mu_0 \in ID(\mathbb{R}^d)$ such that $\mu = \Phi_c(\mu_0)$. There is a finite, right-continuous, decreasing version $G_\xi(u)$ of $G_\xi^*(u)$, and the relation between ν and the polar decomposition $\lambda_0(\mathrm{d}\xi)\nu_{0\xi}(\mathrm{d}u)$*

of the Lévy measure ν_0 of μ_0 is described as follows. We have

$$\nu_0(B) = -c\int_S \lambda(\mathrm{d}\xi)\int_0^\infty 1_B(u\xi)\mathrm{d}G_\xi(u) \text{ for } B\in\mathcal{B}(\mathbb{R}^d), \tag{2.9}$$

$$k_\xi(u) = \frac{\nu_{0\xi}(e^{-c}u,u]}{ca_\xi},\quad G_\xi(u) = \frac{\nu_{0\xi}(u,\infty)}{ca_\xi},\quad \lambda(\mathrm{d}\xi) = a_\xi\lambda_0(\mathrm{d}\xi), \tag{2.10}$$

where

$$a_\xi = \int_0^\infty (1\wedge u^2)\nu_{0\xi}(e^{-c}u,u]\frac{\mathrm{d}u}{cu}. \tag{2.11}$$

Proof. Let (A,ν,γ) be the triplet of μ.

Step 1. Assume that ν satisfies (a)–(b). By Proposition 2.6 we have $G_\xi^*(u)<\infty$ almost surely. Therefore, $G_\xi^*(u)$ has a finite, decreasing, right-continuous version $G_\xi(u)$ by (b). For fixed ξ, $-G_\xi(u)$ induces a σ-finite measure on $(0,\infty)$. We show that the measure ν_0 defined in (2.9) is a Lévy measure and that a distribution $\mu_0\in ID(\mathbb{R}^d)$ satisfies $\mu=\Phi_c(\mu_0)$ if it has triplet (A_0,ν_0,γ_0), where

$$A_0 = \frac{2c}{e^{2c}-1}A,\quad \gamma_0 = \frac{c}{e^c-1}\left(\gamma+\int_{\mathbb{R}^d}\nu_0(\mathrm{d}x)\int_0^1 e^{cs}x1_{\{s:\,e^{-cs}<|x|\leqslant 1\}}(s)\mathrm{d}s\right). \tag{2.12}$$

We have (2.7). Indeed, $G_\xi(e^{-c}u)-G_\xi(u)=k_\xi(u)$ and hence with substitution $u=e^{cs}r$ we get

$$\begin{aligned}
&\int_{\mathbb{R}^d}\nu_0(\mathrm{d}x)\int_0^1 1_B(e^{cs}x)\mathrm{d}s = -c\int_S\lambda(\mathrm{d}\xi)\int_0^\infty \mathrm{d}G_\xi(r)\int_0^1 1_B(e^{cs}r\xi)\mathrm{d}s\\
=\ & -c\int_S\lambda(\mathrm{d}\xi)\int_0^\infty \mathrm{d}G_\xi(r)\int_r^{e^cr}1_B(u\xi)\frac{\mathrm{d}u}{cu}\\
=\ & -\int_S\lambda(\mathrm{d}\xi)\int_0^\infty 1_B(u\xi)\frac{\mathrm{d}u}{u}\int_0^\infty 1_{\{r:\,r\leqslant u<e^cr\}}(r)\mathrm{d}G_\xi(r)\\
=\ & \int_S\lambda(\mathrm{d}\xi)\int_0^\infty 1_B(u\xi)\left(G_\xi(e^{-c}u)-G_\xi(u)\right)\frac{\mathrm{d}u}{u}\\
=\ & \int_S\lambda(\mathrm{d}\xi)\int_0^\infty 1_B(u\xi)\frac{k_\xi(u)}{u}\mathrm{d}u = \nu(B).
\end{aligned}$$

Notice that

$$\int_{\mathbb{R}^d}\left(1\wedge|x|^2\right)\nu_0(\mathrm{d}x)\leqslant\int_{\mathbb{R}^d}\nu_0(\mathrm{d}x)\int_0^1\left(1\wedge|e^{cs}x|^2\right)\mathrm{d}s = \int_{\mathbb{R}^d}\left(1\wedge|x|^2\right)\nu(\mathrm{d}x)<\infty,$$

where the equality is by (2.7). Hence ν_0 is a Lévy measure. Defining A_0 and γ_0 by (2.12) it is readily verified that we have (2.6) and (2.8). Hence the distribution μ_0 with triplet (A_0,ν_0,γ_0) satisfies $\mu=\Phi_c(\mu_0)$.

Step 2. Let $\mu=\Phi_c(\mu_0)$, where $\mu_0\in ID(\mathbb{R}^d)$ has triplet (A_0,ν_0,γ_0). To complete the proof it suffices to show (a)–(b), (2.10)–(2.11), and that (A_0,ν_0,γ_0) satisfies (2.9) and (2.12). (These imply the uniqueness of μ_0, since λ is unique and the function $G_\xi(u)$ is unique for λ-a. e. ξ.) Using (2.7) and substitution $e^{cs}r = u$, we get

$$\begin{aligned}
\nu(B) &= \int_{\mathbb{R}^d} \nu_0(\mathrm{d}x) \int_0^1 1_B(e^{cs}x)\mathrm{d}s = \int_S \lambda_0(\mathrm{d}\xi) \int_0^\infty \nu_{0\xi}(\mathrm{d}r) \int_0^1 1_B(e^{cs}r\xi)\mathrm{d}s \\
&= \int_S \lambda_0(\mathrm{d}\xi) \int_0^\infty \nu_{0\xi}(\mathrm{d}r) \int_r^{e^c r} 1_B(u\xi)\frac{\mathrm{d}u}{cu} \\
&= \int_S \lambda_0(\mathrm{d}\xi) \int_0^\infty 1_B(u\xi)\frac{\nu_{0\xi}(e^{-c}u, u]}{cu}\mathrm{d}u.
\end{aligned}$$

Hence we have (a). By the normalization $\int_0^\infty (1 \wedge r^2)\nu_\xi(\mathrm{d}r) = 1$ we get the representations of $\lambda(\mathrm{d}\xi)$ and $k_\xi(u)$ in (2.10)–(2.11). Since $k_\xi(e^{jc}u) = (ca_\xi)^{-1}\nu_{0\xi}(e^{(j-1)c}u, e^{jc}u]$ we have $\sum_{j=1}^\infty k_\xi(e^{jc}u) = (ca_\xi)^{-1}\nu_{0\xi}(u,\infty)$. Obviously $G_\xi(u) := (ca_\xi)^{-1}\nu_{0\xi}(u,\infty)$ is finite, right-continuous and decreasing. In particular we have (b). The expression (2.12) follows from (2.6) and (2.8). Finally, in order to show (2.9) note that we have

$$-c\int_S \lambda(\mathrm{d}\xi) \int_0^\infty 1_B(u\xi)\mathrm{d}G_\xi(u) = c\int_S \lambda(\mathrm{d}\xi) \int_0^\infty 1_B(u\xi)\frac{\nu_{0\xi}(\mathrm{d}u)}{ca_\xi}$$

$$= \int_S \lambda_0(\mathrm{d}\xi) \int_0^\infty 1_B(u\xi)\nu_{0\xi}(\mathrm{d}u) = \nu_0(B)$$

by (2.10)–(2.11). □

Corollary 2.8. *For $c \in \mathbb{R}\setminus\{0\}$ the mapping $\Phi_c \colon ID(\mathbb{R}^d) \longrightarrow I(c)$ is one-to-one.*

To prove the corollary note that we may assume $c > 0$ by Proposition 2.4 (iv), in which case the result follows from (ii) above. This corollary shows that if $\{X_s^j\}_{s\in[0,1]}$, $j = 1, 2$, are periodic Ornstein-Uhlenbeck processes with parameter $c > 0$ and background driving Lévy processes $\{Z_s^j\}_{s\in[0,1]}$ such that $\mathcal{L}(X_s^1) = \mathcal{L}(X_s^2)$, then $\mathcal{L}(Z_1^1) = \mathcal{L}(Z_1^2)$. That is, the stationary distribution of a periodic Ornstein-Uhlenbeck process determines the law of the background driving Lévy process uniquely.

Remark 2.9. In the next result it is convenient to let $I(0) = ID(\mathbb{R}^d)$. Also let $\Phi_0 \colon I(0) \longrightarrow I(0)$ be the identity mapping.

Proposition 2.10. *Let c_n and c be real numbers with $c_n \to c$.*

(i) *Let $\mu_{n,0} \in ID(\mathbb{R}^d)$ for $n = 1, 2, \dots$ and $\mu_{n,0} \to \mu_0$. Then $\mu_0 \in ID(\mathbb{R}^d)$ and $\Phi_{c_n}(\mu_{n,0}) \to \Phi_c(\mu_0)$.*

(ii) *Conversely, let $\mu_n \in I(c_n)$ and $\mu_n \to \mu$. Then $\mu \in I(c)$ and $\Phi_{c_n}^{-1}(\mu_n) \to \Phi_c^{-1}(\mu)$.*

Before proving this result we need two lemmas.

Lemma 2.11. *Let $c > 0$ and $\mu = \Phi_c(\mu_0)$, where μ and μ_0 have triplets (A,ν,γ) and (A_0, ν_0, γ_0), respectively. Then we have*

$$\int_{|x|>a} \nu_0(\mathrm{d}x) \leqslant \int_{|x|>a} \nu(\mathrm{d}x) \leqslant \int_{|x|>e^{-c}a} \nu_0(\mathrm{d}x), \quad a > 0, \tag{2.13}$$

$$\frac{e^{2c}-1}{2c} \int_{|x|\leqslant e^{-c}} |x|^2 \nu_0(\mathrm{d}x) \leqslant \int_{|x|\leqslant 1} |x|^2 \nu(\mathrm{d}x) \leqslant \frac{e^{2c}-1}{2c} \int_{|x|\leqslant 1} |x|^2 \nu_0(\mathrm{d}x), \tag{2.14}$$

$$\left| \int_{\mathbb{R}^d} \nu_0(\mathrm{d}x) \int_0^1 e^{cs} x 1_{\{x\colon e^{-cs}<|x|\leqslant 1\}}(x) \mathrm{d}s \right| \leqslant \frac{e^c - 1}{c} \int_{e^{-c}<|x|\leqslant 1} \nu_0(\mathrm{d}x). \tag{2.15}$$

Proof. For $a > 0$ let $D^a = \{y \in \mathbb{R}^d\colon |y| > a\}$. Then, for $s \in [0,1]$ we have $1_{D^a}(x) \leqslant 1_{D^a}(e^{cs}x) \leqslant 1_{D^a}(e^c x)$ and hence

$$\int_{D^a} \nu_0(\mathrm{d}x) \leqslant \int_{\mathbb{R}^d} \nu_0(\mathrm{d}x) \int_0^1 1_{D^a}(e^{cs}x)\mathrm{d}s \leqslant \int_{\mathbb{R}^d} 1_{D^a}(e^c x)\nu_0(\mathrm{d}x),$$

from which (2.13) follows since $\int_{\mathbb{R}^d} \nu_0(\mathrm{d}x) \int_0^1 1_{D^a}(e^{cs}x)\mathrm{d}s = \nu(D^a)$ by (2.7). Note that we have

$$\int_{\mathbb{R}^d} f(x)\nu(\mathrm{d}x) = \int_{\mathbb{R}^d} \nu_0(\mathrm{d}x) \int_0^1 f(e^{cs}x)\mathrm{d}s \tag{2.16}$$

for all nonnegative measurable functions f by (2.7). Thus (2.14) follows. Since $|x|1_{\{e^{-cs}<|x|\leqslant 1\}}(x) \leqslant 1_{\{e^{-c}<|x|\leqslant 1\}}(x)$ for $0 < s < 1$, (2.15) is obvious. □

Remark 2.12. Let $c > 0$ and let μ, μ_0, ν, ν_0 be as in Lemma 2.11. We can also prove the following from (2.16).

(i) $\nu(\mathbb{R}^d) < \infty$ if and only if $\nu_0(\mathbb{R}^d) < \infty$.
(ii) Let $0 < \alpha < 2$. Then $\int_{|x|\leqslant 1} |x|^\alpha \nu(\mathrm{d}x) < \infty$ if and only if $\int_{|x|\leqslant 1} |x|^\alpha \nu_0(\mathrm{d}x) < \infty$.
(iii) Let $\alpha > 0$. Then $\int_{|x|>1} |x|^\alpha \nu(\mathrm{d}x) < \infty$ if and only if $\int_{|x|>1} |x|^\alpha \nu_0(\mathrm{d}x) < \infty$.
(iv) Let $\alpha > 0$. Then $\int_{|x|>1} (\log|x|)^\alpha \nu(\mathrm{d}x) < \infty$ if and only if $\int_{|x|>1} (\log|x|)^\alpha \nu_0(\mathrm{d}x) < \infty$.

Recall that a set of probability measures on $\mathbb{R}^d$ is precompact if and only if any sequence in the set has a subsequence which converges weakly.

Lemma 2.13. *Let M be a subclass of $ID(\mathbb{R}^d)$ and let C be a bounded set in $\mathbb{R}$. If M is precompact then $\{\Phi_c(\mu_0)\colon \mu_0 \in M,\ c \in C\}$ and $\{\Phi_c^{-1}(\mu)\colon \mu \in M,\ c \in C\}$ are both precompact.*

Proof. Let N denote a subset of $ID(\mathbb{R}^d)$ and $\mu \in N$ have triplet $(A_\mu, \nu_\mu, \gamma_\mu)$. Then, N is precompact if and only if the following four conditions (i)–(iv) are satisfied, where (i) $\sup_{\mu\in N} \sup_{|x|\leqslant 1} |A_\mu x| < \infty$; (ii) $\sup_{\mu\in N} \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu_\mu(\mathrm{d}x) < \infty$; (iii) $\sup_{\mu\in N} |\gamma_\mu| < \infty$;

(iv) $\lim_{l\to\infty}\sup_{\mu\in N}\nu_\mu(\{|x|>l\})=0$. Indeed, this is essentially E 12.5 in [12] except that (iv) is missing in that exercise. Hence, we have to show that (i)–(iv) are satisfied with $N=\{\Phi_c(\mu_0)\colon \mu_0\in M,\ c\in C\}$ and $\{\Phi_c^{-1}(\mu)\colon \mu\in M,\ c\in C\}$ if (i)–(iv) are satisfied with $N=M$. This is, however, easily verified using (2.6)–(2.8) and Lemma 2.11 if $C\subseteq[0,\infty)$. Further, use Proposition 2.4 (iv) if $C\cap(-\infty,0)\neq\emptyset$. $\square$

Proof of Proposition 2.10. (i) By [12], Lemma 7.8, $\mu_0\in ID(\mathbb{R}^d)$. Let $\Psi_0(z)=\log\widehat{\mu}_0(z)$ and $\Psi_{n,0}(z)=\log\widehat{\mu_{n,0}}(z)$. By Lemma 7.7 in [12], $\Psi_{n,0}(z)\to\Psi_0(z)$ uniformly on any compact set. Hence $\int_0^1\Psi_{n,0}(e^{c_n s}z)\mathrm{d}s\to\int_0^1\Psi_0(e^{cs}z)\mathrm{d}s$ for all $z\in\mathbb{R}^d$. Hence, by (2.5) $\widehat{\Phi_{c_n}(\mu_{n,0})}(z)\to\widehat{\Phi_c(\mu_0)}(z)$. Note that (2.5) is trivially true for $c=0$.

(ii) $\{\mu_n\colon n=1,2,\ldots\}$ is precompact. Hence $\{\Phi_{c_n}^{-1}(\mu_n)\colon n=1,2,\ldots\}$ is precompact by Lemma 2.13. Thus, there is a subsequence $\{\Phi_{c_{n'}}^{-1}(\mu_{n'})\}$ such that $\Phi_{c_{n'}}^{-1}(\mu_{n'})\to\mu_0$ where μ_0 is a probability measure on $\mathbb{R}^d$. We have $\mu_0\in ID(\mathbb{R}^d)$ and by (i) $\mu_{n'}\to\Phi_c(\mu_0)$. Hence $\mu_0=\Phi_c^{-1}(\mu)$. Thus μ_0 is independent of the choice of subsequence. Hence $\Phi_{c_n}^{-1}(\mu_n)\to\Phi_c^{-1}(\mu)$. $\square$

3. Classes $I(c)$ for varying values of c

In this section we study the relations between classes $I(c)$ for different values of c. First we compare $I(c)$ to $SD(\mathbb{R}^d)$.

Theorem 3.1. (i) *We have $SD(\mathbb{R}^d)\subsetneqq I(c)$ for any $c\in\mathbb{R}\setminus\{0\}$.*

(ii) *Let $c_n\in\mathbb{R}$ with $c_n\uparrow\infty$. Then $\bigcap_n I(c_n)=SD(\mathbb{R}^d)$.*

Remark 3.2. Recall that a distribution is selfdecomposable if and only if it is the stationary distribution of a usual Ornstein-Uhlenbeck process (driven by a Lévy process). Hence the result in (i) shows that a stationary distribution of a usual Ornstein-Uhlenbeck process is, for any $c\neq 0$, the stationary distribution of a periodic Ornstein-Uhlenbeck process with parameter c as well.

Proof of Theorem 3.1. (i) We may and do assume $c>0$ without loss of generality. Let $\mu\in SD(\mathbb{R}^d)$. Then μ satisfies condition (a) of Theorem 2.7 and $k_\xi(u)$ is decreasing in u, see [12], Theorem 15.10. Hence also (b) of Theorem 2.7 is satisfied, which means $\mu\in I(c)$. If $\mu\in SD(\mathbb{R}^d)$ its Lévy measure has total mass 0 or ∞. Hence Remark 2.12 (i) implies $SD(\mathbb{R}^d)\subsetneqq I(c)$.

(ii) *Step* 1. Let $\mu = \mathcal{L}\left(\int_0^c e^{-s}\mathrm{d}Z_s^{(c)}\right)$ with $0 < c < \infty$, where $\{Z_s^{(c)}\}_{s\geqslant 0}$ is a Lévy process on $\mathbb{R}^d$. Then the Lévy measures ν and $\nu^{(c)}$ of μ and $\mathcal{L}(Z_1^{(c)})$ are related as

$$\nu(B) = \int_{\mathbb{R}^d} \nu^{(c)}(\mathrm{d}x) \int_0^c 1_B(e^{-s}x)\mathrm{d}s, \qquad B \in \mathcal{B}(\mathbb{R}^d). \tag{3.1}$$

Let $\lambda^{(c)}(\mathrm{d}\xi)\nu_\xi^{(c)}(\mathrm{d}r)$ be the polar decomposition of $\nu^{(c)}$. As in the second step of the proof of Theorem 2.7 one shows that

$$\nu(B) = \int_S \lambda^{(c)}(\mathrm{d}\xi) \int_0^\infty 1_B(u\xi)\nu_\xi^{(c)}(u, e^c u]\frac{\mathrm{d}u}{u}. \tag{3.2}$$

Notice that if $B \subseteq \{x \colon \alpha < |x| \le \beta\}$ with $0 < \alpha < \beta < \infty$, then

$$\begin{aligned} &\int_S \lambda^{(c)}(\mathrm{d}\xi) \int_0^\infty 1_B(u\xi)\nu_\xi^{(c)}(u,\infty)\frac{\mathrm{d}u}{u} \leqslant \int_S \lambda^{(c)}(\mathrm{d}\xi) \int_\alpha^\beta \nu_\xi^{(c)}(u,\infty)\frac{\mathrm{d}u}{u} \\ \leqslant\ & \frac{\beta-\alpha}{\alpha} \int_S \lambda^{(c)}(\mathrm{d}\xi)\nu_\xi^{(c)}(\alpha,\infty) < \infty. \end{aligned}$$

It follows from (3.2) that

$$\nu(B) = \int_S \lambda^{(c)}(\mathrm{d}\xi) \int_0^\infty 1_B(u\xi)\nu_\xi^{(c)}(u,\infty)\frac{\mathrm{d}u}{u} - \int_S \lambda^{(c)}(\mathrm{d}\xi) \int_0^\infty 1_B(u\xi)\nu_\xi^{(c)}(e^c u,\infty)\frac{\mathrm{d}u}{u} \tag{3.3}$$

if $B \subseteq \{x \colon \alpha < |x| \leqslant \beta\}$ for some $0 < \alpha < \beta < \infty$.

Step 2. Let c, ν and $\nu^{(c)}$ be as in Step 1. For any $l > 0$ we have $\int_{|x|>l} \nu(\mathrm{d}x) \geqslant c\int_{|x|>e^c l} \nu^{(c)}(\mathrm{d}x)$. Indeed, using (3.2) and $D^l = \{y \colon |y| > l\}$, we get

$$\begin{aligned} \int_{|x|>l} \nu(\mathrm{d}x) &= \int_{\mathbb{R}^d} \nu^{(c)}(\mathrm{d}x) \int_0^c 1_{D^l}(e^{-s}x)\mathrm{d}s = c\int_{\mathbb{R}^d} \nu^{(c)}(\mathrm{d}x) \int_0^1 1_{D^l}(e^{-cs}x)\mathrm{d}s \\ &= c\int_{\mathbb{R}^d} \nu^{(c)}(\mathrm{d}x) \int_0^1 1_{D^{le^c}}(e^{cs}x)\mathrm{d}s \geqslant c\int_{\mathbb{R}^d} 1_{D^{le^c}}(x)\nu^{(c)}(\mathrm{d}x). \end{aligned}$$

Step 3. Assume that $\mu \in \bigcap_{n=1}^\infty I(c_n)$ for some $c_n \uparrow \infty$. Since $I(c_n) = I(-c_n)$, μ has the representation $\mu = \mathcal{L}\left(\int_0^{c_n} e^{-s}\mathrm{d}Z_s^{(c_n)}\right)$ by Remark 2.2. Let $B \in \mathcal{B}(\mathbb{R}^d)$ satisfying $B \subseteq \{x \colon \alpha < |x| \leqslant \beta\}$ with $0 < \alpha < \beta < \infty$. By Step 2 we see that

$$\begin{aligned} &\int_S \lambda^{(c_n)}(\mathrm{d}\xi) \int_0^\infty 1_B(u\xi)\nu_\xi^{(c_n)}(e^{c_n}u,\infty)\frac{\mathrm{d}u}{u} \leqslant \int_S \lambda^{(c_n)}(\mathrm{d}\xi) \int_\alpha^\beta \nu_\xi^{(c_n)}(e^{c_n}u,\infty)\frac{\mathrm{d}u}{u} \\ \leqslant\ & \frac{\beta-\alpha}{\alpha} \int_S \lambda^{(c_n)}(\mathrm{d}\xi)\nu_\xi^{(c_n)}(e^{c_n}\alpha,\infty) = \frac{\beta-\alpha}{\alpha}\nu^{(c_n)}\left(\{x \colon |x| > e^{c_n}\alpha\}\right) \\ \leqslant\ & \frac{\beta-\alpha}{\alpha c_n}\nu\left(\{x \colon |x| > \alpha\}\right) \to 0 \qquad \text{as } n \to \infty. \end{aligned}$$

Hence, by (3.3),

$$\nu(B) = \lim_{n\to\infty} \int_S \lambda^{(c_n)}(\mathrm{d}\xi) \int_0^\infty 1_B(u\xi)\nu_\xi^{(c_n)}(u,\infty)\frac{\mathrm{d}u}{u}. \tag{3.4}$$

It follows that, for $b > 1$,

$$\begin{aligned}\nu(b^{-1}B) &= \lim_{n\to\infty} \int_S \lambda^{(c_n)}(\mathrm{d}\xi) \int_0^\infty 1_B(bu\xi)\nu_\xi^{(c_n)}(u,\infty)\frac{\mathrm{d}u}{u}\\ &= \lim_{n\to\infty} \int_S \lambda^{(c_n)}(\mathrm{d}\xi) \int_0^\infty 1_B(u\xi)\nu_\xi^{(c_n)}(b^{-1}u,\infty)\frac{\mathrm{d}u}{u}.\end{aligned}$$

Since $\nu_\xi^{(c_n)}(b^{-1}u,\infty) \geqslant \nu_\xi^{(c_n)}(u,\infty)$, we get $\nu(b^{-1}B) \geqslant \nu(B)$ for every $B \in \mathcal{B}(\mathbb{R}^d)$ satisfying $B \subseteq \{x\colon \alpha < |x| \leqslant \beta\}$ with $0 < \alpha < \beta < \infty$. This shows that μ is selfdecomposable by [12], Theorem 15.8. □

For $\mu \in ID(\mathbb{R}^d)$ let us introduce $Q(\mu)$, the set of $c \geqslant 0$ for which μ is in $I(c)$. That is, $Q(\mu) = \{c \geqslant 0\colon \mu \in I(c)\}$. Here we let $I(0) = ID(\mathbb{R}^d)$ as in Remark 2.9. Hence $0 \in Q(\mu)$. Due to Proposition 2.4 (iv) we consider only nonnegative values of c.

Proposition 3.3. *Let $\mu \in ID(\mathbb{R}^d)$.*

(i) *The set $Q(\mu)$ is a closed subset of $\mathbb{R}_+$;*
(ii) *if $c \in Q(\mu)$ then $c/n \in Q(\mu)$ for all $n \in \mathbb{N}$;*
(iii) *the set $Q(\mu)$ is either equal to $[0,\infty)$ or bounded;*
(iv) *we have $Q(\mu) = [0,\infty)$ if and only if $\mu \in SD(\mathbb{R}^d)$.*

Proof. The fact (i) follows from Proposition 2.10 while (iii) and (iv) are due to Theorem 3.1; (ii) is equivalent to the assertion that $I(c) \subseteq I(c/n)$ for $n \in \mathbb{N}$. Assume that $c > 0$ and $\mu \in I(c)$. Let us prove that $\mu \in I(c')$ for $c' = c/n$. Denote $G^*_{\xi,l}(u) = \sum_{j=1}^\infty k_\xi\left(e^{(jn-l)c'}u\right)$ for $l = 0,1,\dots,n-1$, where k_ξ is defined in Theorem 2.7. By Theorem 2.7 $G^*_{\xi,l}(u)$ has a decreasing version. Since $G^*_{\xi,0}(u)+\cdots+G^*_{\xi,n-1}(u) = \sum_{j=1}^\infty k_\xi(e^{jc'}u)$ we have $\mu \in I(c')$, using Theorem 2.7 again. □

Let us consider some special distributions to show the rich variety of the set $Q(\mu)$.

Proposition 3.4. *Let $d = 1$.*

(i) *Let μ be an infinitely divisible distribution $\mathbb{R}$ for which the Lévy measure is not absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. Then $Q(\mu) = \{0\}$.*

In the following let $\mu \in ID(\mathbb{R})$ have Lévy measure ν given by $\nu(\mathrm{d}u) = 1_{(0,\infty)}(u)\frac{k(u)}{u}\mathrm{d}u$. Let $l(v) = k(e^v)$ for $v \in \mathbb{R}$.

(ii) *If $c > 0$ and $\mu = \Phi_c(\mu_0)$, where μ_0 is the Poisson distribution with mean 1, then $l(v) = \frac{1}{c}1_{[0,c)}(v)$ and $Q(\mu) = \{0\} \cup \{c/n\colon n = 1, 2, \ldots\}$.*

(iii) *Assume $l(v)$ is either strictly increasing or strictly decreasing on an interval $[a, b)$ with $a < b$ and that $l(v) = 0$ for $v \in \mathbb{R} \setminus [a, b)$. Then $Q(\mu) = \{0\}$.*

(iv) *Let $l(v)$ be decreasing on $(-\infty, 0]$ with $l(0) = 2$ and*

$$l(v) = \begin{cases} v + 2, & v \in [0, 1] \\ 3 - 3(v - 1), & v \in [1, 2] \\ 0, & v \geqslant 2. \end{cases} \tag{3.5}$$

Then $Q(\mu) = [0, 1]$.

(v) *Let $l(v)$ be decreasing on $(-\infty, 0]$ with $l(0) = 4$ and*

$$l(v) = \begin{cases} 4 + 2v, & v \in [0, 1] \\ 6 - 6(v - 1), & v \in [1, 3/2] \\ 3 - 3(v - 3/2), & v \in [3/2, 5/2] \\ 0, & v \geqslant 5/2. \end{cases} \tag{3.6}$$

Then $Q(\mu) = [0, 3/4] \cup [1, 3/2]$.

Proof. (i) For $c > 0$ a necessary condition for a distribution to be in $I(c)$ is by Theorem 2.7 that its Lévy measure is absolutely continuous.

To prove (ii)–(v) let $M_c(v) = \sum_{j=0}^{\infty} l(v + jc)$ for $c > 0$ and $v \in \mathbb{R}$.

(ii) The Lévy measure of μ_0 is $\nu_0 = \delta_1$, which by Theorem 2.7 implies that the Lévy measure of μ is $\nu(\mathrm{d}u) = 1_{(0,\infty)}(u)\frac{k(u)}{u}\mathrm{d}u$, where $k(u) = \frac{1}{c}1_{[1,e^c)}(u)$. Hence $l(v) = \frac{1}{c}1_{[0,c)}(v)$. From Proposition 3.3 (ii) it follows that $Q(\mu)$ includes $\{0\} \cup \{c/n\colon n = 1, 2, \ldots\}$. Let $c' \in (0, \infty) \setminus \{c/n\colon n = 1, 2, \ldots\}$. In order to show $c' \notin Q(\mu)$ it suffices by Theorem 2.7 to show that $M_{c'}(v)$ does not have a decreasing version on $\mathbb{R}$. If $c' > c$, then $M_{c'}(v)$ is 0 on $[c - c', 0)$ and $1/c$ on $[0, c)$, which means $c' \notin Q(\mu)$. If $0 < c' < c$ with $c' \notin \{c/n\colon n = 1, 2, \ldots\}$, then choosing $n \in \mathbb{N}$ such that $c/(n + 1) < c' < c/n$, we see that $M_{c'}(v)$ equals n/c on $(c - (n+1)c', 0)$ and $(n+1)/c$ on $(0, c - nc')$, which means $c' \notin Q(\mu)$.

(iii) First we consider the case where $l(v)$ is strictly increasing on $[a, b)$. Let $c > 0$. By Theorem 2.7 we have to show that $M_c(v)$ does not have a decreasing version. If $a \vee (b - c) < v_1 < v_2 < b$ then $M_c(v_1) = l(v_1) < l(v_2) = M(v_2)$ and the result follows.

Next assume that $l(v)$ is strictly decreasing on $[a,b)$. It is readily seen that $M_c(v)$ is strictly decreasing on $[a,b)$. For $v<a$ we have

$$M_c(v) = \sum_{j=0}^{\infty} l(v+jc) = \sum_{j=1}^{\infty} l(v+jc) = M_c(v+c).$$

Let $c>0$. If $v_1 < a < v_2 < v_1 + c < b$, then $M_c(v_1) = M_c(v_1+c) < M_c(v_2)$. Hence $c \notin Q(\mu)$.

(iv) It suffices to show that

(a) $v \mapsto M_c(v)$ is decreasing for $c \in [1/2, 1]$,

(b) $v \mapsto M_c(v)$ does not have a decreasing version for $c > 1$.

Indeed, by Theorem 2.7 and Proposition 3.3 (ii), (a) and (b) imply that $c \notin Q(\mu)$ for $c>1$ and $c/n \in Q(\mu)$ for $c \in [1/2,1]$ and $n \in \mathbb{N}$. It follows that $Q(\mu) = [0,1]$.

Proof of (a). Let $c \in [1/2,1]$. We consider four cases.

Case 1: $v \in [1,\infty)$. Note that $v \mapsto l(v+jc)$ is decreasing on $[1,\infty)$ for $j \geqslant 0$. Thus $M_c(v)$ is decreasing in v for $v \in [1,\infty)$.

Case 2: $v \in [1-c, 1]$. Similar to case 3 below and hence omitted.

Case 3: $v \in [0, 1-c]$. Let $0 \leqslant v_1 \leqslant v_2 \leqslant 1-c$. Then $0 \leqslant v_1 + c \leqslant v_2 + c \leqslant 1$, $1 \leqslant v_1 + 2c \leqslant v_2 + 2c \leqslant 2$ and $1 \leqslant v_1 + jc \leqslant v_2 + jc$ for $j \geqslant 2$. Hence $l(v_2) - l(v_1) = l(v_2+c) - l(v_1+c) = v_2 - v_1$, $l(v_2+2c) - l(v_1+2c) = -3(v_2 - v_1)$ and $l(v_2+jc) - l(v_1+jc) \leqslant 0$ for $j \geqslant 2$. Therefore $M_c(v_2) - M_c(v_1) \leqslant -(v_2 - v_1) \leqslant 0$.

Case 4: $v \in (-\infty, 0]$. Let v_1, v_2 satisfy $-c \leqslant v_1 \leqslant v_2 \leqslant 0$. Then $l(v_1) \geqslant l(v_2)$ since $l(v)$ is decreasing on $(-\infty, 0]$, and $M_c(v_1+c) \geqslant M_c(v_2+c)$ since $0 \leqslant v_1 + c \leqslant v_2 + c$ and $M_c(v)$ is decreasing on $[0,\infty)$. Since $M_c(v) = l(v) + M_c(v+c)$, we have $M_c(v_1) \geqslant M_c(v_2)$, which shows that $M_c(v)$ is decreasing on $[-c, 0]$. Repeating inductively it follows that $M_c(v)$ is decreasing on any interval $[-cp, -c(p-1)]$ for $p \in \mathbb{N}$.

Proof of (b). Let $c>1$. If $2-c < v_1 < v_2 < 1$, then $M_c(v_1) = l(v_1) < l(v_2) = M_c(v_2)$.

(v) We show the following.

(c) $v \mapsto M_c(v)$ is decreasing for $c \in [1/4, 1/2]$.

(d) $v \mapsto M_c(v)$ does not have a decreasing version for $c \in (3/4, 1)$.

(e) $v \mapsto M_c(v)$ is decreasing for $c \in [1, 3/2]$.

(f) $v \mapsto M_c(v)$ does not have a decreasing version for $c > 3/2$.

Since $c \in Q(\mu)$ implies $c/n \in Q(\mu)$ for all $n \in \mathbb{N}$, it follows from (c) and (e) that $[0, 1/2]$ and $[1/2, 3/4]$ are subsets of $Q(\mu)$. Hence, it follows from (c)–(f) that $Q(\mu) = [0, 3/4] \cup [1, 3/2]$.

Proof of (c). Fix $c \in [1/4, 1/2]$. Clearly $M_c(v)$ is decreasing on $[1, \infty)$. Let $q_0 \in \{2, 3, 4\}$ be the least q such that $qc \geqslant 1$. We prove that $M_c(v)$ is decreasing on the interval $[(1 - qc) \vee 0, 1 - (q - 1)c]$ for $q = 1, \ldots, q_0$. Since $(1 - q_0 c) \vee 0 = 0$, this implies that $M_c(v)$ is decreasing on $[0, 1]$, and (c) follows from decreasingness of $l(v)$ on $(-\infty, 0]$. Hence let $q \in \{1, \ldots, q_0\}$ and $v_1, v_2 \in [(1 - qc) \vee 0, 1 - (q - 1)c]$ with $v_1 \leqslant v_2$. For $p = 0, \ldots, q - 1$ and $i = 1, 2$ we have $v_i + pc \in [0, 1]$. Hence $l(v_1 + pc) - l(v_2 + pc) = 2(v_1 - v_2)$. Since $v_i + qc \in [1, 3/2]$ and $v_i + (q + 1)c \in [1, 5/2]$ it follows that $l(v_1 + qc) - l(v_2 + qc) = -6(v_1 - v_2)$ and $l(v_1 + (q + 1)c) - l(v_2 + (q + 1)c) \geqslant -3(v_1 - v_2)$. Moreover, $l(v_1 + jc) \geqslant l(v_2 + jc)$ for $j \geqslant q + 2$. Hence $M_c(v_1) - M_c(v_2) \geqslant -(v_1 - v_2)$. That is, $M_c(v_1) \geqslant M_c(v_2)$, which means that $M_c(v)$ is decreasing on the interval $[1 - qc, 1 - (q - 1)c]$.

Proof of (d). Fix $c \in (3/4, 1)$. Consider v_1 and v_2 satisfying $(7/4 - 2c) \vee 0 < v_1 < v_2 < 1 - c$. Then $7/4 < v_i + 2c < 1 + c < 2$ and $v_i + 3c > 5/2$ for $i = 1, 2$. Hence $l(v_1) - l(v_2) = 2(v_1 - v_2)$, $l(v_1 + c) - l(v_2 + c) = 2(v_1 - v_2)$, $l(v_1 + 2c) - l(v_2 + 2c) = -3(v_1 - v_2)$, and $l(v_1 + jc) - l(v_2 + jc) = 0$ for $j \geqslant 3$. Hence $M_c(v_1) - M_c(v_2) = v_1 - v_2 < 0$, which gives (d).

Proof of (e). Fix $c \in [1, 3/2]$. Clearly $M_c(v)$ is decreasing on $[1, \infty)$. Let $v_1, v_2 \in [0, 1]$ with $v_1 \leqslant v_2$. Then $v_i + c \in [1, 5/2]$ and $v_i + jc \geqslant 2$ for $i = 1, 2$ and $j \geqslant 2$. Hence $l(v_1) - l(v_2) = 2(v_1 - v_2)$, $l(v_1 + c) - l(v_2 + c) \geqslant -3(v_1 - v_2)$ and $M_c(v_1) - M_c(v_2) \geqslant -(v_1 - v_2) \geqslant 0$. Hence $M_c(v)$ is decreasing on $[0, \infty)$. This yields (e) as in case 4 of (a).

Proof of (f). Consider v_1 and v_2 satisfying $(5/2 - 2c) \vee 0 < v_1 < v_2 < 1$. Then $M_c(v_1) = l(v_1) < l(v_2) = M_c(v_2)$. Hence we have (f). □

Proposition 3.5. *Let* $0 < c' < c$.

(i) $I(c) \subsetneqq I(c')$ *if* $c/c' \in \mathbb{N}$,

(ii) $I(c) \not\subseteq I(c')$ *and* $I(c) \not\supseteq I(c')$ *if* $c/c' \notin \mathbb{N}$.

Proof. The fact that $I(c) \subseteq I(c')$ for $c/c' \in \mathbb{N}$ is already proved in Proposition 3.3 (ii). To show (ii) and the strict inclusion in (i) we may and do assume $d = 1$. (Indeed, by considering Lévy measures on $\mathbb{R}^d$ in polar form $\lambda(\mathrm{d}\xi)\nu_\xi(\mathrm{d}u)$ where λ is a point measure, the general case follows easily from the case $d = 1$.) Let μ_0 be a Poisson distribution. Let $\mu_1 = \Phi_{c'}(\mu_0)$ and $\mu_2 = \Phi_c(\mu_0)$. Then $Q(\mu_1) = \{0\} \cup \{c'/n \colon n \in \mathbb{N}\}$

and $Q(\mu_2) = \{0\} \cup \{c/n \colon n \in \mathbb{N}\}$ by Proposition 3.4. Thus, $\mu_1 \in I(c') \setminus I(c)$ and for $c/c' \notin \mathbb{N}$ we have $\mu_2 \in I(c) \setminus I(c')$. $\square$

Remark 3.6. Let c_n and c be nonnegative with $c_n \to c$. Assume $c_n \neq c$ for all n. Then, as a complement to Theorem 3.1, we have

$$SD(\mathbb{R}^d) \subsetneqq \bigcap_{n \geqslant 1} I(c_n) \subsetneqq I(c). \tag{3.7}$$

Here we let $I(0) = ID(\mathbb{R}^d)$ as in Remark 2.9. The first inclusion in (3.7) follows since $SD(\mathbb{R}^d) \subseteq I(c')$ for all c'. To see that it is strict choose $\mu \in ID(\mathbb{R}^d)$ with $Q(\mu) = [0, c+k]$ where $k > 0$ is such that $c_n \leqslant c + k$ for all n. Such a μ exists by a straightforward extension of Proposition 3.4 (iv). Hence, $\mu \in I(c_n)$ for all n and $\mu \notin SD(\mathbb{R}^d)$.

Let $\mu \in \bigcap_{n \geqslant 1} I(c_n)$. Then $c_n \in Q(\mu)$ for all n. Since $Q(\mu)$ is closed by Proposition 3.3 it follows that $c \in Q(\mu)$. To see that the last inclusion is strict let us consider the cases $c = 0$ and $c > 0$ separately. If $c = 0$ then $I(c) = ID(\mathbb{R}^d)$. Hence, choosing $\mu \in ID(\mathbb{R}^d)$ such that $\mu \notin I(c')$ for any $c' > 0$ it follows that the inclusion is strict. In the case $c > 0$ we can take $\mu \in ID(\mathbb{R}^d)$ such that $Q(\mu) = \{0\} \cup \{c/n \colon n = 1, 2, \ldots\}$. Then $\mu \in I(c) \setminus \bigcap_{n \geqslant 1} I(c_n)$.

4. Relations to other classes of distributions

We use the following notation as in [10]. For $\alpha \in (0,2]$ let $\mathfrak{S}_\alpha(\mathbb{R}^d)$ be the set of α-stable distributions on $\mathbb{R}^d$. That is, a distribution μ on $\mathbb{R}^d$ is in $\mathfrak{S}_\alpha(\mathbb{R}^d)$ if and only if for every $n \in \mathbb{N}$ there is a $\beta \in \mathbb{R}^d$ such that $\widehat{\mu}(z)^n = \widehat{\mu}(n^{1/\alpha} z) e^{\mathrm{i}\langle \beta, z\rangle}$. (The definition of stability of μ is different from [12], p. 76, when μ is trivial.) Then $\mathfrak{S}(\mathbb{R}^d) = \bigcup_{\alpha \in (0,2]} \mathfrak{S}_\alpha(\mathbb{R}^d)$ is the set of stable distributions. For any distinct α and α' in $(0,2]$ $\mathfrak{S}_\alpha(\mathbb{R}^d) \cap \mathfrak{S}_{\alpha'}(\mathbb{R}^d)$ is exactly the class of trivial distributions. Let $\mathfrak{S}^0_\alpha(\mathbb{R}^d)$ be the set of distributions μ on $\mathbb{R}^d$ such that for every $n \in \mathbb{N}$ $\widehat{\mu}(z)^n = \widehat{\mu}(n^{1/\alpha} z)$. Then $\mathfrak{S}^0(\mathbb{R}^d) = \bigcup_{\alpha \in (0,2]} \mathfrak{S}^0_\alpha(\mathbb{R}^d)$ is the set of strictly stable distributions. For any distinct α and α' in $(0,2]$ $\mathfrak{S}^0_\alpha(\mathbb{R}^d) \cap \mathfrak{S}^0_{\alpha'}(\mathbb{R}^d)$ consists of the single element δ_0. Distributions δ_c with $c \neq 0$ belong to $\mathfrak{S}^0_1(\mathbb{R}^d)$ but does not belong to any $\mathfrak{S}^0_\alpha(\mathbb{R}^d)$ with $\alpha \neq 1$.

Let $\{X_s\}_{s \in [0,1]}$ be a periodic Ornstein-Uhlenbeck process with parameter $c > 0$ and background driving Lévy process $\{Z_s\}_{s \in [0,1]}$. Consider the following problem: To what extent are stability and the class L_m-property inherited from $\mathcal{L}(Z_1)$ to the stationary distribution $\mathcal{L}(X_s)$? We can recast this as the question to what extent Φ_c preserves stability and the class L_m-property.

Proposition 4.1. *Let $c \in \mathbb{R} \setminus \{0\}$, $\mu_0 \in ID(\mathbb{R}^d)$ and let $\mu = \Phi_c(\mu_0)$. Let $\alpha \in (0,2]$. Then*

(i) *$\mu \in \mathfrak{S}^0_\alpha(\mathbb{R}^d)$ if and only if $\mu_0 \in \mathfrak{S}^0_\alpha(\mathbb{R}^d)$;*

(ii) *$\mu \in \mathfrak{S}_\alpha(\mathbb{R}^d)$ if and only if $\mu_0 \in \mathfrak{S}_\alpha(\mathbb{R}^d)$.*

Proof. Let Ψ_0 be the characteristic exponent of $\widehat{\mu}_0$. (i) Assume $\mu_0 \in \mathfrak{S}^0_\alpha(\mathbb{R}^d)$. Then $a\Psi_0(z) = \Psi_0(a^{1/\alpha}z)$ for all $a > 0$. This implies

$$\widehat{\mu}(z)^a = \exp \int_0^1 a\Psi_0(e^{cs}z)\mathrm{d}s = \exp \int_0^1 \Psi_0(e^{cs}a^{1/\alpha}z)\mathrm{d}s = \widehat{\mu}(a^{1/\alpha}z).$$

Hence $\mu \in \mathfrak{S}^0_\alpha(\mathbb{R}^d)$. Conversely, if $\mu \in \mathfrak{S}^0_\alpha(\mathbb{R}^d)$ then $\widehat{\mu}(z)^a = \widehat{\mu}(a^{1/\alpha}z)$ for all $a > 0$. Then by Corollary 2.8 we have $a\Psi_0(z) = \Psi_0(a^{1/\alpha}z)$, which implies $\mu_0 \in \mathfrak{S}^0_\alpha(\mathbb{R}^d)$.

Assume $\mu_0 \in \mathfrak{S}_\alpha(\mathbb{R}^d)$. Then for any $a > 0$ there is $\gamma_a \in \mathbb{R}^d$ such that $a\Psi_0(z) = \Psi_0(a^{1/\alpha}z) + \mathrm{i}\langle \gamma_a, z\rangle$. Then

$$\begin{aligned}\widehat{\mu}(z)^a &= \exp \int_0^1 a\Psi_0(e^{cs}z)\mathrm{d}s = \exp\left(\int_0^1 \Psi_0(e^{cs}a^{1/\alpha}z)\mathrm{d}s + \mathrm{i}\int_0^1 \langle \gamma_a, e^{cs}z\rangle \mathrm{d}s\right)\\ &= \widehat{\mu}(a^{1/\alpha}z)e^{\mathrm{i}c^{-1}(e^c-1)\langle \gamma_a, z\rangle}.\end{aligned}$$

Hence $\mu \in \mathfrak{S}_\alpha(\mathbb{R}^d)$ and conversely. $\square$

Let us recall the definition of the classes $L_m(\mathbb{R}^d)$ for $m = 0, 1, \ldots$. Let $L_0(\mathbb{R}^d) = SD(\mathbb{R}^d)$. That is, μ is in $L_0(\mathbb{R}^d)$ if for every $b > 1$ there is a distribution μ_b such that

$$\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\mu}_b(z). \tag{4.1}$$

Sometimes μ_b is referred to as an innovation distribution generated by μ. For $m = 1, 2, \ldots$, $L_m(\mathbb{R}^d)$ is the class of distributions μ such that, for every $b > 1$, there exists $\mu_b \in L_{m-1}(\mathbb{R}^d)$ satisfying (4.1). We have

$$L_0(\mathbb{R}^d) \supsetneqq L_1(\mathbb{R}^d) \supsetneqq L_2(\mathbb{R}^d) \supsetneqq \cdots \supsetneqq L_\infty(\mathbb{R}^d) \supsetneqq \mathfrak{S}(\mathbb{R}^d), \tag{4.2}$$

where $L_\infty(\mathbb{R}^d) = \bigcap_{0 \leqslant m < \infty} L_m(\mathbb{R}^d)$. These sets were introduced by Urbanik [15], [16] and further developed by Sato [11] in connection to limit distribution theory for sums of independent random vectors.

Remark 4.2. We recall a convenient characterization of $L_m(\mathbb{R}^d)$.

(i) Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$. For $\epsilon > 0$ let Δ_ϵ be the difference operator $\Delta_\epsilon f(v) = f(v+\epsilon) - f(v)$, and Δ^n_ϵ be the nth iteration of Δ_ϵ. Hence,

$$\Delta^n_\epsilon f(v) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(v + j\epsilon).$$

Let $\Delta_\epsilon^0 f = f$. We say that f is monotone of order n if $\Delta_\epsilon^j f \geqslant 0$ for all $\epsilon > 0$ and $j = 0, 1, \ldots, n$. From [10], Lemma 18, or [11], Lemma 3.2, we know that f is monotone of order $n \geqslant 2$ if and only if $f \in C^{n-2}$, $f^{(j)} \geqslant 0$ for $j = 0, 1, \ldots, n-2$, and $f^{(n-2)}$ is increasing and convex.

(ii) Let $\mu \in ID(\mathbb{R}^d)$ have Lévy measure ν with polar decomposition $\lambda(\mathrm{d}\xi)\nu_\xi(\mathrm{d}u)$. By [12], Theorem 15.8, $\mu \in L_0(\mathbb{R}^d) = SD(\mathbb{R}^d)$ if and only if $\nu(b^{-1}B) \geqslant \nu(B)$ for $B \in \mathcal{B}(\mathbb{R}^d)$ and $b > 1$. This is the case if and only if, for λ-a.e. ξ, $\nu_\xi(b^{-1}B) \geqslant \nu_\xi(B)$ for $B \in \mathcal{B}((0, \infty))$ and $b > 1$. The latter is equivalent to $\nu_\xi(b^{-1}a_1, b^{-1}a_2] \geqslant \nu_\xi(a_1, a_2]$ for $0 < a_1 < a_2$ and $b > 1$, which is equivalent to the property that $\nu_\xi(e^{-v-\epsilon}, e^{-v}]$ is increasing in v for every $\epsilon > 0$. If $\mu \in L_0(\mathbb{R}^d)$ then $\nu_\xi(\mathrm{d}u) = k_\xi(u)\mathrm{d}u/u$ with k_ξ being decreasing, and conversely.

(iii) Let $\mu \in L_0(\mathbb{R}^d)$. That is, μ is selfdecomposable. Hence, the Lévy measure of μ is decomposed in polar form as $\lambda(\mathrm{d}\xi)k_\xi(u)\mathrm{d}u/u$, where $k_\xi(u)$ is decreasing. Let $h_\xi(v) = k_\xi(e^{-v})$. Let $m \in \{1, 2, \ldots\}$. From [10], Theorem 20, or [11], Theorems 3.2 and 3.3, we know that μ is in $L_m(\mathbb{R}^d)$ if and only if h_ξ is monotone of order $m+1$ for λ-a.e. ξ.

Remark 4.3. Let $c > 0$. For $\mu_0 \in ID_{\log}(\mathbb{R}^d)$ we define $\widetilde{\Phi}_{-c}(\mu_0) = \mathcal{L}\left(\int_0^\infty e^{-ct}\mathrm{d}Z_t\right)$ where $\{Z_t\}_{t\geqslant 0}$ is a Lévy process with $\mathcal{L}(Z_1) = \mu_0$. Then $\widetilde{\Phi}_{-c}$ is a one-to-one mapping from $ID_{\log}(\mathbb{R}^d)$ onto $L_0(\mathbb{R}^d)$, as is explained in Section 1 in connection to usual Ornstein-Uhlenbeck processes driven by Lévy processes. Let $m \in \{-1, 0, 1, \ldots\}$ and $j \in \{1, 2, \ldots\}$. The following facts are known: the class of μ_0 for which the jth iteration $\widetilde{\Phi}_{-c}^j$ of $\widetilde{\Phi}_{-c}$ is definable coincides with $ID_{\log^j}(\mathbb{R}^d)$; $\mu \in L_{m+j}(\mathbb{R}^d)$ if and only if $\mu = \widetilde{\Phi}_{-c}^j(\mu_0)$ with some $\mu_0 \in L_m(\mathbb{R}^d) \cap ID_{\log^j}(\mathbb{R}^d)$ where we let $L_{-1}(\mathbb{R}^d) = ID(\mathbb{R}^d)$. See [5], [10] (Theorems 46 and 49), [14].

Proposition 4.4. *Let $M \subseteq ID(\mathbb{R}^d)$ and $c \in \mathbb{R} \setminus \{0\}$. Assume that M is completely closed and satisfies* (P)*. Then $\Phi_c(M) = \Phi_{-c}(M) \subseteq M$ and $\Phi_c(M)$ is completely closed and satisfies* (P)*. Moreover, we have $\Phi_{nc}(M) \subseteq \Phi_c(M)$ for all $n \in \mathbb{N}$. In addition, if M satisfies* (D) *then $\Phi_c(M)$ satisfies* (D)*; if M satisfies* (GJ) *then $\Phi_c(M)$ satisfies* (GJ)*.*

Proof. Assume M is completely closed and satisfies (P). Let $\mu_0 \in M$. Then $\Phi_c(\mu_0) = \mathcal{L}(\int_0^1 e^{cs}\mathrm{d}Z_s)$, where $\{Z_s\}_{s\geqslant 0}$ is a Lévy process with $\mathcal{L}(Z_1) = \mu_0$. There is a sequence of positive step functions $f_1, f_2, \ldots$ such that

$$\int_0^1 e^{cs}\mathrm{d}Z_s = \lim_{n\to\infty}\int_0^1 f_n(s)\mathrm{d}Z_s,$$

where the limit is in probability. There are $0 = s_{n,0} < s_{n,1} < \ldots < s_{n,k} = 1$ and $a_{n,j} > 0$ such that

$$\int_0^1 f_n(s)\mathrm{d}Z_s = \sum_{j=1}^{k} a_{n,j}\left(Z_{s_{n,j}} - Z_{s_{n,j-1}}\right).$$

The distribution of the right-hand side belongs to M by (C1), (C3) and (P). Hence $\Phi_c(\mu_0) \in M$ by (C2). This proves $\Phi_c(M) \subseteq M$.

To show $\Phi_c(M) = \Phi_{-c}(M)$, let $\mu = \Phi_c(\mu_0)$ with $\mu_0 \in M$. Let $\mu' = \Phi_{-c}(\mu_0)$. By Proposition 2.4 (iv), $\widehat{\mu'}(z) = \widehat{\mu}(e^{-c}z)$, that is, $\widehat{\mu'}(e^c z) = \widehat{\mu}(z)$. Hence $\mu \in \Phi_{-c}(M)$ by (C3) and $\Phi_c(M) \subseteq \Phi_{-c}(M)$. Changing the role of c and $-c$, we get the converse inclusion.

It follows from Proposition 2.10 that $\Phi_c(M)$ satisfies (C2). It is moreover easily verified that $\Phi_c(M)$ satisfies (C1) and (C3), so let us verify (P) for $\Phi_c(M)$. Let $\mu = \mathcal{L}(\int_0^1 e^{cs}\mathrm{d}Z_s)$, where $\mathcal{L}(Z_1) \in M$. We have $\mu^t = \mathcal{L}(\int_0^1 e^{cs}\mathrm{d}_s Z_{ts})$. Since $\mathcal{L}(Z_{ts}) \in M$ for all $t, s \geqslant 0$ by (P) of M it follows that $\mu^t \in \Phi_c(M)$.

Assume $\mu \in \Phi_{nc}(M)$ and let $\{Z_s\}_{s\geqslant 0}$ be a Lévy process with $\mu = \mathcal{L}\left(\int_0^1 e^{ncs}\mathrm{d}Z_s\right)$. By Remark 2.2 we have

$$\begin{aligned}\int_0^1 e^{ncs}\mathrm{d}Z_s &= \int_0^n e^{cs}\mathrm{d}_s Z_{s/n} = \sum_{j=0}^{n-1}\int_j^{j+1} e^{cs}\mathrm{d}_s Z_{s/n}\\ &= \sum_{j=0}^{n-1} e^{jc}\int_j^{j+1} e^{c(s-j)}\mathrm{d}_s Z_{s/n} \stackrel{\mathrm{d}}{=} \sum_{j=0}^{n-1} e^{jc}\int_0^1 e^{cs}\mathrm{d}_s Z^j_{s/n},\end{aligned}$$

where $\{Z^j_s\}_{s\geqslant 0}$, $j = 0, \ldots, n-1$, are independent copies of $\{Z_s\}_{s\geqslant 0}$ and $\stackrel{\mathrm{d}}{=}$ denotes equality in distribution. We have $\mathcal{L}(Z_u) \in M$ for all $u \geqslant 0$ by (P) of M. Hence $\mathcal{L}\left(\int_0^1 e^{cs}\mathrm{d}Z^j_{s/n}\right) \in \Phi_c(M)$ and thus $\mu = \mathcal{L}\left(\sum_{j=0}^{n-1} e^{jc}\int_0^1 e^{cs}\mathrm{d}Z^j_{s/n}\right) \in \Phi_c(M)$ since $\Phi_c(M)$ is completely closed.

If M satisfies (D) then it is easy to see that $\Phi_c(M)$ satisfies (D).

Assume that M satisfies (GJ). Let $\mu = \Phi_c(\mu_0)$ with $\mu_0 \in M$ and $\mu = \mu_G * \mu_J$ where μ_G and μ_J are, respectively, Gaussian and purely non-Gaussian. We have $\mu_0 = \mu_{0G} * \mu_{0J}$ with μ_{0G} Gaussian and μ_{0J} purely non-Gaussian. Then $\mu = \Phi_c(\mu_0) = \Phi_c(\mu_{0G}) * \Phi_c(\mu_{0J})$ by Proposition 2.4 (ii). By Proposition 2.3 $\Phi_c(\mu_{0G})$ is Gaussian and $\Phi_c(\mu_{0J})$ is purely non-Gaussian. It follows that there is $b \in \mathbb{R}^d$ such that $\mu_G =$

$\Phi_c(\mu_{0G}) * \delta_b$ and $\mu_J = \Phi_c(\mu_{0J}) * \delta_{-b}$. Since $\mu_{0G}, \mu_{0J} \in M$ by (GJ) of M we get $\mu_G, \mu_J \in \Phi_c(M)$ by (C3) of $\Phi_c(M)$. $\square$

As an immediate consequence of Proposition 4.4 we have the following.

Corollary 4.5. *Let $c \in \mathbb{R} \setminus \{0\}$, $m = 0, 1, \ldots, \infty$ and $n \in \mathbb{N}$. Then $\Phi_c(L_m(\mathbb{R}^d))$ is completely closed and satisfies* (D), (P) *and* (GJ), *and $\Phi_{nc}(L_m(\mathbb{R}^d)) \subseteq \Phi_c(L_m(\mathbb{R}^d)) = \Phi_{-c}(L_m(\mathbb{R}^d)) \subseteq L_m(\mathbb{R}^d)$.*

Indeed, $L_m(\mathbb{R}^d)$ is completely closed and satisfies (D), (P) and (GJ).

Theorem 4.6. *Let $c \in \mathbb{R} \setminus \{0\}$. Then,*
(i) *$L_0(\mathbb{R}^d) \subsetneqq I(c)$;*
(ii) *$L_{m+1}(\mathbb{R}^d) \subsetneqq \Phi_c(L_m(\mathbb{R}^d)) \subsetneqq L_m(\mathbb{R}^d)$ for $m = 0, 1, \ldots$;*
(iii) *$L_\infty(\mathbb{R}^d) = \Phi_c(L_\infty(\mathbb{R}^d))$.*

Remark 4.7. Using Proposition 4.4 and (ii) above it is immediate that $\Phi_c(L_{m+1}(\mathbb{R}^d)) \subsetneqq \Phi_c^2(L_m(\mathbb{R}^d)) \subsetneqq \Phi_c(L_m(\mathbb{R}^d))$ for $c \neq 0$ and $m \geqslant 0$. We do not know the relation between $\Phi_c^2(L_m(\mathbb{R}^d))$ and $L_{m+1}(\mathbb{R}^d)$.

Proof of Theorem 4.6. The assertion (i) is by Theorem 3.1.

(ii) The second inclusion is by Corollary 4.5. Let $\mu \in L_{m+1}(\mathbb{R}^d)$ with $m \geqslant 0$. There exists by (i) a distribution $\mu_0 \in ID(\mathbb{R}^d)$ such that $\mu = \Phi_c(\mu_0)$. We show that $\mu_0 \in L_m(\mathbb{R}^d)$, which yields the first inclusion. Decompose the Lévy measure of μ as $\lambda(\mathrm{d}\xi)k_\xi(u)\mathrm{d}u/u$ in polar form and let $h_\xi(v) = k_\xi(e^{-v})$ for $v \in \mathbb{R}$. Let ν_0 denote the Lévy measure of μ_0.

Step 1. First we prove $\mu_0 \in L_0(\mathbb{R}^d)$. (This concludes the proof when $m = 0$.) Define $H_\xi(v) := \sum_{j=1}^\infty k_\xi(e^{jc-v}) = \sum_{j=1}^\infty h_\xi(v - jc)$ for $v \in \mathbb{R}$. Since by Theorem 2.7 we have $H_\xi(v) = (ca_\xi)^{-1}\nu_{0\xi}(e^{-v}, \infty)$ it follows that

$$\begin{aligned} & \sum_{j=1}^\infty \Delta_\epsilon h_\xi(v - jc) = \Delta_\epsilon H_\xi(v) \\ = \quad & (ca_\xi)^{-1}\left(\nu_{0\xi}(e^{-v-\epsilon}, \infty) - \nu_{0\xi}(e^{-v}, \infty)\right) = (ca_\xi)^{-1}\nu_{0\xi}(e^{-v-\epsilon}, e^{-v}]. \end{aligned}$$

This implies that $\nu_{0\xi}(e^{-v-\epsilon}, e^{-v}]$ is increasing in v for $\epsilon > 0$, since $h_\xi(v)$ is increasing and convex by Remark 4.2. Hence $\mu_0 \in L_0(\mathbb{R}^d)$ by the same remark.

Step 2. Assume $m \geqslant 1$. Since $\mu_0 \in L_0(\mathbb{R}^d)$ the polar decomposition of ν_0 is $\lambda_0(\mathrm{d}\xi)k_{0\xi}(u)\mathrm{d}u/u$, where the function $h_{0\xi}(v) := k_{0\xi}(e^{-v})$ is nonnegative and increasing. By Remark 4.2, in order to show $\mu_0 \in L_m(\mathbb{R}^d)$ it suffices to show that $h_{0\xi}(v)$ is monotone of order $m+1$. Since $\mu \in L_{m+1}(\mathbb{R}^d)$ the same remark shows that $h_\xi(v)$ is monotone of order $m+2$ and hence that $h'_\xi(w) = \frac{\mathrm{d}}{\mathrm{d}w} h_\xi(w)$ exists and is monotone of order $m+1$. Define $H_\xi(v)$ as above. Since $h_\xi(-\infty) = 0$ and $h'_\xi(w) \geqslant 0$ we have

$$H_\xi(v) = \sum_{j=1}^{\infty} \int_{-\infty}^{v-jc} h'_\xi(w)\mathrm{d}w = \sum_{j=1}^{\infty} \int_{-\infty}^{v} h'_\xi(w-jc)\mathrm{d}w = \int_{-\infty}^{v} \left(\sum_{j=1}^{\infty} h'_\xi(w-jc) \right) \mathrm{d}w.$$

Hence $H_\xi(v)$ is of class C^1 with $\frac{\mathrm{d}}{\mathrm{d}v} H_\xi(v) = \sum_{j=1}^{\infty} h'_\xi(v-jc)$. On the other hand,

$$H_\xi(v) = (ca_\xi)^{-1} \nu_{0\xi}(e^{-v}, \infty) = (ca_\xi)^{-1} \int_{e^{-v}}^{\infty} k_{0\xi}(u) \frac{\mathrm{d}u}{u}$$

and thus $\frac{\mathrm{d}}{\mathrm{d}v} H_\xi(v) = (ca_\xi)^{-1} k_{0\xi}(e^{-v}) = (ca_\xi)^{-1} h_{0\xi}(v)$. It follows that $(ca_\xi)^{-1} h_{0\xi}(v) = \sum_{j=1}^{\infty} h'_\xi(v-jc)$ and

$$(ca_\xi)^{-1} \left(\Delta_\epsilon^i h_{0\xi} \right)(v) = \sum_{j=1}^{\infty} \left(\Delta_\epsilon^i h'_\xi \right)(v-jc) \geqslant 0 \text{ for } i = 0, 1, \ldots, m+1,$$

completing the proof of the first inclusion. We will show in Examples 4.8–4.9 below that the two inclusions are strict.

(iii) We have from the first inclusion in (ii) and from the one-to-one property of Φ_c that

$$L_\infty(\mathbb{R}^d) = \bigcap_{0 \leqslant m < \infty} L_{m+1}(\mathbb{R}^d) \subseteq \bigcap_{0 \leqslant m < \infty} \Phi_c(L_m(\mathbb{R}^d)) = \Phi_c(L_\infty(\mathbb{R}^d)).$$

On the other hand, $\Phi_c(L_\infty(\mathbb{R}^d)) \subseteq L_\infty(\mathbb{R}^d)$ by Corollary 4.5. $\square$

Example 4.8. We show that the second inclusion in Theorem 4.6 (ii) is strict. We may and do assume $d = 1$ and using Corollary 4.5 it suffices to consider the case $c > 0$. Example (i) shows that the inclusion is strict when $m = 0$, while example (ii) applies to the case $m \geqslant 1$.

(i) We construct $\mu \in L_0(\mathbb{R})$ such that $\mu_0 = \Phi_c^{-1}(\mu) \notin L_0(\mathbb{R})$. Let $\mu \in ID(\mathbb{R})$ have Lévy measure given by $\nu(\mathrm{d}u) = k(u) 1_{(0,\infty)}(u)\mathrm{d}u/u$, where

$$k(u) = \begin{cases} 1 & \text{for } 0 < u \leqslant 1 \\ u^{-\alpha} & \text{for } u > 1 \end{cases}$$

with $\alpha > 0$. Then $h(v) = k(e^{-v})$ equals 1 for $v \geqslant 0$ and $e^{\alpha v}$ for $v \leqslant 0$. This h is increasing but not convex. Hence μ is in $L_0(\mathbb{R})$ but not in $L_1(\mathbb{R})$. Let $\mu_0 = \Phi_c^{-1}(\mu)$, and denote the Lévy measure of μ_0 by ν_0. As in the proof of Theorem 4.6 we have $H(v+\epsilon) - H(v) = \frac{1}{c}\nu_0(e^{-v-\epsilon}, e^{-v}]$, where $H(v) = \sum_{j=1}^{\infty} h(v-jc)$. Since $H(v) = e^{\alpha(v-c)}/(1-e^{-\alpha c})$ for $v \leqslant c$ and $H(v) = 1 + e^{\alpha(v-2c)}/(1-e^{-\alpha c})$ for $c \leqslant v \leqslant 2c$, we can check that $H'(c-) > H'(c+)$, which shows that $H(v)$ is not convex. Thus for some $\epsilon > 0$ $\nu_0(e^{-v-\epsilon}, e^{-v}] = c(H(v+\epsilon) - H(v))$ is not increasing in v. Therefore $\mu_0 \notin L_0(\mathbb{R})$.

(ii) For $\mu \in L_1(\mathbb{R})$ we have $\mu_0 = \Phi_c^{-1}(\mu) \in L_0(\mathbb{R})$ by Theorem 4.6. Let the Lévy measures of μ and μ_0 be, respectively, $\nu(\mathrm{d}u) = 1_{(0,\infty)}(u)k(u)\mathrm{d}u/u$ and $\nu_0(\mathrm{d}u) = 1_{(0,\infty)}(u)k_0(u)\mathrm{d}u/u$. Defining $h(v) = k(e^{-v})$ and $h_0(v) = k_0(e^{-v})$, Theorem 2.7 shows that $h(v) = c^{-1}\int_v^{v+c} h_0(w)\mathrm{d}w$. Hence

$$h'(v) = c^{-1}(h_0(v+c) - h_0(v)) \tag{4.3}$$

if h_0 is continuous.

Let $m \geqslant 1$. We construct $\mu_0 \in L_0(\mathbb{R})$ such that

(a) $h_0(v)$ is continuous and monotone of order m but not of order $m+1$,

(b) $h_0(v+c) - h_0(v)$ is monotone of order m,

which by (4.3) and Remark 4.2 shows that $\mu_0 \in L_{m-1}(\mathbb{R}) \setminus L_m(\mathbb{R})$ and $\mu = \Phi_c(\mu_0) \in L_m(\mathbb{R})$.

Let $0 < a < 2$, $a < b$ and $b \leqslant ae^{ac}$. Define $g_j(v)$ for $j = 0, \dots, m-1$ as

$$\begin{aligned} g_0(v) &= \begin{cases} e^{bv} & \text{for } v \leqslant 0 \\ e^{av} & \text{for } v \geqslant 0, \end{cases} \\ g_j(v) &= \int_{-\infty}^{v} g_{j-1}(w)\mathrm{d}w, \quad \text{for } j = 1, \dots, m-1. \end{aligned}$$

Using that $0 < a < 2$ and $b > 0$, we see that $h_0(v) := g_{m-1}(v)$ satisfies the integrability condition

$$\int_{-\infty}^{0} h_0(v)\mathrm{d}v + \int_0^\infty e^{-2v}h_0(v)\mathrm{d}v < \infty$$

in [11], p. 215, which implies that ν_0 constructed from $h_0(v)$ is indeed a Lévy measure. Since $g_0(v)$ is nonnegative, continuous and increasing but not convex, we have (a) by Remark 4.2. Using that $0 < b \leqslant ae^{ac}$ and $a < b$, it is readily verified that $g_0(v+c) - g_0(v)$ is increasing and nonnegative. When $m = 1$, this fact gives (b). Assume $m \geqslant 2$. Let $f(v) = h_0(v+c) - h_0(v)$ and notice that

$$f^{(j)}(v) = \int_{-\infty}^{v} (g_{m-2-j}(w+c) - g_{m-2-j}(w))\mathrm{d}w, \quad j = 0, \dots, m-2. \tag{4.4}$$

By (4.4) this implies that $f^{(m-2)}$ is increasing, nonnegative and convex, which in turn implies that $f^{(j)}$, $j = 0, \dots, m-2$, have the same properties. Now (b) follows from Remark 4.2.

Example 4.9. We show that the first inclusion in Theorem 4.6 (ii) is strict. It suffices to give an example of a distribution in $\Phi_{-c}(L_m(\mathbb{R}^d)) \setminus L_{m+1}(\mathbb{R}^d)$ in the case $c > 0$ and $d = 1$.

Let $b = e^c$. By Proposition 5.1 in Appendix there is $\mu \in L_{m+1}(\mathbb{R})$ such that $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\mu}_b(z)$ with $\mu_b \in L_m(\mathbb{R}) \setminus L_{m+1}(\mathbb{R})$. By Remark 4.3, there is $\mu_0 \in L_m(\mathbb{R}) \cap ID_{\log}(\mathbb{R})$ such that $\mu = \widetilde{\Phi}_{-c}(\mu_0)$. Let $\{Z_s\}_{s\geqslant 0}$ be a Lévy process with $\mathcal{L}(Z_1) = \mu_0$. Since

$$\int_0^\infty e^{-cs}\mathrm{d}Z_s = \int_1^\infty e^{-cs}\mathrm{d}Z_s + \int_0^1 e^{-cs}\mathrm{d}Z_s \text{ and } \int_1^\infty e^{-cs}\mathrm{d}Z_s \stackrel{\mathrm{d}}{=} e^{-c}\int_0^\infty e^{-cs}\mathrm{d}Z_s,$$

we have $\mu_b = \Phi_{-c}(\mu_0)$. Hence $\Phi_{-c}(L_m(\mathbb{R}))$ contains $\mu_b \in L_m(\mathbb{R}) \setminus L_{m+1}(\mathbb{R})$.

Remark 4.10. For $m \in \{0, 1, \ldots, \infty\}$ let M_m be the class of probability measures ρ on $\mathbb{R}^d$ such that

$$\widehat{\rho}(z) = \frac{\widehat{\mu}(z)}{\widehat{\mu}(b^{-1}z)} \tag{4.5}$$

for some $\mu \in L_m(\mathbb{R}^d)$ and some $b > 1$. Thus, ρ is an innovation distribution generated by a selfdecomposable distribution $\mu \in L_m(\mathbb{R}^d)$. Characterization of this class M_m is of some interest. Let us see that

$$M_m = \bigcup_{c>0} \Phi_c(L_{m-1}(\mathbb{R}^d)) \cap ID_{\log}(\mathbb{R}^d)$$

where $L_{-1}(\mathbb{R}^d) = ID(\mathbb{R}^d)$ and $L_{\infty-1}(\mathbb{R}^d) = L_\infty(\mathbb{R}^d)$. Indeed, let $\rho \in M_m$ be given by (4.5) with $\mu \in L_m(\mathbb{R}^d)$. Let $c = \log b$. Then $\mu = \widetilde{\Phi}_{-c}(\mu_0)$ for some $\mu_0 \in ID_{\log}(\mathbb{R}^d) \cap L_{m-1}(\mathbb{R}^d)$ by Remark 4.3. Repeating the arguments in Example 4.9 it follows that $\rho = \Phi_{-c}(\mu_0) \in \Phi_{-c}(L_{m-1}(\mathbb{R}^d)) = \Phi_c(L_{m-1}(\mathbb{R}^d))$. Since $\mu_0 \in ID_{\log}(\mathbb{R}^d)$ we have also $\rho \in ID_{\log}(\mathbb{R}^d)$ by Remark 2.12 (iv).

For the converse, let $\rho \in \Phi_c(L_{m-1}(\mathbb{R}^d)) \cap ID_{\log}(\mathbb{R}^d)$ with some $c > 0$. Noting that $\Phi_c(L_{m-1}(\mathbb{R}^d)) = \Phi_{-c}(L_{m-1}(\mathbb{R}^d))$, let $\mu_0 = \Phi_{-c}^{-1}(\rho)$. Then $\mu_0 \in L_{m-1}(\mathbb{R}^d)$ and since $\rho \in ID_{\log}(\mathbb{R}^d)$ we have also $\mu_0 \in ID_{\log}(\mathbb{R}^d)$ by Remark 2.12 (iv). Let $\mu = \widetilde{\Phi}_{-c}(\mu_0)$. Then $\mu \in L_m(\mathbb{R}^d)$ by Remark 4.3 and $\widehat{\mu}(z) = \widehat{\rho}(z)\widehat{\mu}(e^{-c}z)$. Therefore $\rho \in M_m$.

5. Appendix

Recall that, for $m = 0, 1, \ldots$, $\mu \in L_m(\mathbb{R}^d)$ if and only if there exists for every $b > 1$ a distribution $\mu_b \in L_{m-1}(\mathbb{R}^d)$ such that we have (4.1). (Here we let $L_{-1}(\mathbb{R}^d) = ID(\mathbb{R}^d)$).

Proposition 5.1. *Let $m \in \{0, 1, \dots\}$.*

(a) *There is a distribution $\mu \in L_m(\mathbb{R}^d) \setminus L_{m+1}(\mathbb{R}^d)$ such that, for every $b > 1$, $\mu_b \in L_{m-1}(\mathbb{R}^d) \setminus L_m(\mathbb{R}^d)$.*

(b) *Let $b > 1$. Then there exists $\mu \in L_m(\mathbb{R}^d) \setminus L_{m+1}(\mathbb{R}^d)$ such that $\mu_b \in L_m(\mathbb{R}^d)$.*

Proof. Assume without loss of generality $d = 1$. Let $\mu \in L_0(\mathbb{R})$ have Lévy measure ν given by $\nu(\mathrm{d}u) = 1_{(0,\infty)}(u)k(u)\mathrm{d}u/u$. Let $h(v) = k(e^{-v})$ for $v \in \mathbb{R}$. For $b > 1$ the Lévy measure ν_b of μ_b is $\nu_b(\mathrm{d}u) = 1_{(0,\infty)}(u)k_b(u)\mathrm{d}u/u$, where $k_b(u) = k(u) - k(bu)$. Letting $h_b(v) = k_b(e^{-v})$ we thus have $h_b(v) = h(v) - h(v - \log b)$.

(a) Let μ be the exponential distribution with mean 1. By [12], p. 45, we have $h(v) = e^{-e^{-v}}$. Hence $h''(v) = e^{-e^{-v}}(e^{-2v} - e^{-v}) < 0$ for $v > 0$. Since

$$h_b'(v) = h'(v) - h'(v - \log b) = \int_{v-\log b}^{v} h''(r)\mathrm{d}r < 0 \qquad \text{if } v - \log b > 0,$$

it follows from Remark 4.2 that $\mu_b \in ID(\mathbb{R}) \setminus L_0(\mathbb{R})$ for every $b > 1$. Thus $\mu \in L_0(\mathbb{R}) \setminus L_1(\mathbb{R})$. We have $\mu \in ID_{\log^m}(\mathbb{R})$ and hence $\mu_b \in ID_{\log^m}(\mathbb{R})$ for all m. Fix $c > 0$. Let $\sigma = \widetilde{\Phi}^m_{-c}(\mu)$ and $\sigma_b = \widetilde{\Phi}^m_{-c}(\mu_b)$. Then $\widehat{\sigma}(z) = \widehat{\sigma}(b^{-1}z)\widehat{\sigma}_b(z)$. Observe that $\sigma \in L_m(\mathbb{R}) \setminus L_{m+1}(\mathbb{R})$ and $\sigma_b \in L_{m-1}(\mathbb{R}) \setminus L_m(\mathbb{R})$ by Remark 4.3. As an alternative example, we can use the distribution with Lévy measure $1_{(0,1)}(u)\mathrm{d}u/u$ instead of the exponential distribution.

(b) Given $b > 1$, consider $\mu \in ID(\mathbb{R}) \setminus L_0(\mathbb{R})$ such that $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\mu}_b(z)$ with $\mu_b \in ID(\mathbb{R})$. Such a μ exists due to the existence of a non-selfdecomposable distribution μ that is semi-selfdecomposable with span b, as in [12], Example 15.9. Further we can assume $\mu \in \bigcap_{1\leqslant m<\infty} ID_{\log^m}(\mathbb{R})$ by truncating its Lévy measure or by considering a semi-stable distribution. Fix $c > 0$ and let $\sigma = \widetilde{\Phi}^{m+1}_{-c}(\mu)$. We have $\widehat{\sigma}(z) = \widehat{\sigma}(b^{-1}z)\widehat{\sigma}_b(z)$ with $\sigma_b = \widetilde{\Phi}^{m+1}_{-c}(\mu_b)$. Then, by Remark 4.3, $\sigma \in L_m(\mathbb{R}) \setminus L_{m+1}(\mathbb{R})$ and $\sigma_b \in L_m(\mathbb{R})$. $\square$

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