The Lévy-Itô Decomposition in Free Probability

O.E. Barndorff-Nielsen*† and S. Thorbjørnsen†‡§

Abstract

In this paper we prove the free analog of the Lévy-Itô decomposition for Lévy processes. A significant part of the proof consists of introducing free Poisson random measures, proving their existence and developing a theory of integration w.r.t. such measures. The existence of free Poisson random measures also yields, via the free Lévy-Itô decomposition, an alternative proof of the general existence of free Lévy processes (in law).

1 Introduction.

One of the most fundamental results in the theory of infinitely divisible probability measures (in classical probability) is the Lévy-Khintchine representation, which asserts that a (Borel) probability measure on $\mathbb{R}$ is infinitely divisible, if and only if its cumulant transform has a certain integral representation (cf. Subsection 2.1 below for the precise statement). Historically, though, the Lévy-Khintchine representation was preceded by its counterpart for stochastic processes. Indeed, Paul Lévy established, in the course of proving the Lévy-Khintchine representation, a decomposition of any Lévy process into the sum of two independent processes; a Brownian motion and a process of pure jump type. This decomposition was later proved in full rigour by K. Itô and is now known as the Lévy-Itô decomposition. The Lévy-Itô decomposition is, from the probabilistic point of view, even more fundamental than the Lévy-Khintchine representation.

In our previous paper, [BT], we initiated a study of Lévy processes (in law) in the context of free probability. Our approach was based on the striking correspondence between infinitely divisible probability measures in classical and free probability, respectively, which is known as the Bercovici-Pata bijection. This bijection yields, in particular, a one to one correspondence between classical and free Lévy processes (in law), which makes it possible to export several techniques and results for classical Lévy processes to the free setting. In the present paper, we explore further this technique in establishing the free

*Department of Mathematical Sciences, University of Aarhus, Denmark.

†MaPhySto - The Danish National Research Foundation Network in Mathematical Physics and Stochastics.

‡Department of Mathematics and Computer Science, University of Southern Denmark.

§Supported by the Danish Natural Science Research Council.

version of the Lévy-Itô decomposition for free Lévy processes (in law). Whereas the classical Lévy-Itô decomposition is in the "almost sure sense", we concentrate, in the present note, exclusively on establishing an "in law" version of the free Lévy-Itô decomposition. This is partly due to the techniques we presently have at hand, but it also reflects that a possible analog of the almost sure result is not crucial in the free setting, as long as there is no obvious notion of sample paths for free Lévy processes.

In order to give precise meaning to the pure jump part of the classical Lévy-Itô decomposition of a Lévy process, one needs to introduce the concept of Poisson random measures and to develop integration w.r.t. such measures. The first step towards establishing the free Lévy-Itô decomposition is, similarly, to introduce a notion of free Poisson random measures and to verify their existence. This is done in Section 3 below. In Section 4 we develop the integration theory for free Poisson random measures, which is needed for establishing the free Lévy-Itô decomposition. The integrals (w.r.t. free Poisson random measures) appearing in the free Lévy-Itô decomposition are generally obtained as the limit, in probability (see Subsection 2.3 below), of integrals of simple functions. As a consequence, we need some auxiliary results about the interplay between convergence in probability and free independence. These results are established in Section 5. In Section 6 we prove, after some final preparations, the free version of the Lévy-Itô decomposition. We end the paper by mentioning how the proof of the free Lévy-Itô decomposition together with the existence of free Poisson random measures yields, as in the classical case, the general existence of free Lévy processes (in law), which has previously been noted by Biane (cf. [Bi]) and Voiculescu (cf. [Vo3]). In Section 2, we provide background material on free probability, unbounded operators and the classical Lévy-Itô decomposition.

2 Preliminaries.

In this section we review briefly, for the readers convenience, background material on free probability, unbounded operators and the classical Lévy-Itô decomposition. For a more detailed introduction to free probability and unbounded operators, we refer to Section 2 of [BT].

2.1 Free Probability.

With the work of Schürmann, Speicher, Ben Ghorbal and Muraki, it has quite recently become clear that there are only five "natural" notions of "independence" among (generalised) random variables in quantum probability (see e.g. [Sc],[Sp],[BGS], [Mu]). Among these, the most important and well-studied is, of course, the classical notion of independence (corresponding to tensor products of probability spaces). Since Voiculescu's founding work in the early 1980's, *free independence*, a second of the five types of independences, has been extensively studied; largely due to its applications in operator algebra theory, but also with the purpose of developing a new kind of probability theory parallel to the classical one. Free independence is a purely non-commutative concept, in the sense that it only occurs, except for trivial cases, among non-commuting random variables. These are generally modelled by (selfadjoint) operators on a Hilbert space, belonging to some "non-commutative probability space". Throughout this paper, we shall, partly for the sake of convenience, only deal with the two nicest types of non-commutative probability spaces:

2.1 Definition. (i) A C^*-probability space is a pair $(\mathcal{A}, \phi)$, where $\mathcal{A}$ is a unital C^*-algebra and ϕ is a state on $\mathcal{A}$.

(ii) A W^*-probability space is a pair $(\mathcal{A}, \tau)$, where $\mathcal{A}$ is a von Neumann algebra (acting on some Hilbert space $\mathcal{H}$) and τ is a faithful, normal tracial state on $\mathcal{A}$.

The definition of free independence may then be formulated as follows

2.2 Definition. Let $(\mathcal{A}, \phi)$ be a C^*-probability space.

(i) If $a_1, a_2, \ldots, a_r$ are operators in $\mathcal{A}$, we say that they are *freely independent* w.r.t. ϕ, if

$$\phi\big\{[f_1(a_{i_1}) - \phi(f_1(a_{i_1}))][f_2(a_{i_2}) - \phi(f_2(a_{i_2}))]\cdots[f_p(a_{i_p}) - \phi(f_p(a_{i_p}))]\big\} = 0,$$

for any p in $\mathbb{N}$, any polynomials $f_1, f_2, \ldots, f_p$ in $\mathbb{C}[X]$ and any indices $i_1, i_2, \ldots, i_p$ in $\{1, 2, \ldots, r\}$ satisfying that $i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{p-1} \neq i_p$.

(ii) More generally, a family $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r$ of unital subalgebras of $\mathcal{A}$ is called freely independent w.r.t. ϕ, if, for any indices $i_1, i_2, \ldots, i_p$ in $\{1, 2, \ldots, r\}$ such that $i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{p-1} \neq i_p$, and any operators $a_1 \in \mathcal{A}_{i_1}, a_2 \in \mathcal{A}_{i_2}, \ldots, a_p \in \mathcal{A}_{i_p}$, we have that

$$\phi\big\{[a_1 - \phi(a_1)\mathbf{1}_{\mathcal{A}}][a_2 - \phi(a_2)\mathbf{1}_{\mathcal{A}}]\cdots[a_p - \phi(a_p)\mathbf{1}_{\mathcal{A}}]\big\} = 0.$$

(iii) A family $S_1, S_2, \ldots, S_r$ of subsets of $\mathcal{A}$ is called freely independent, if the unital subalgebras $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r$ of $\mathcal{A}$, generated by $S_1, S_2, \ldots, S_r$, respectively, are freely independent.

Regarding part (i) of the above definition, we may, if $a_1, a_2, \ldots, a_r$ are selfadjoint, consider general continuous functions $f_1, f_2, \ldots, f_r \colon \mathbb{R} \to \mathbb{R}$, instead of polynomials, without changing the definition. (If $(\mathcal{A}, \phi)$ is a W^*-probability space, we may even consider bounded Borel functions instead). The operators $f_j(a_{i_j})$ are then defined in terms of the functional calculus for selfadjoint operators in a C^*-algebra. If a is a selfadjoint operator in a C^*-probability space $(\mathcal{A}, \phi)$, then the functional calculus also gives rise to a distribution of a w.r.t. the state ϕ. Indeed, there exists a unique probability measure μ_a on $\mathbb{R}$, satisfying that

$$\int_{\mathbb{R}} f(t)\, \mu_a(\mathrm{d}t) = \phi(f(a)),$$

for any bounded Borel function $f \colon \mathbb{R} \to \mathbb{R}$. This measure is termed the (spectral) distribution of a w.r.t. τ, and we denote it also by $L\{a\}$. The measure $L\{a\}$ is always concentrated on the spectrum $\mathrm{sp}(a)$ and hence, in particular, compactly supported.

If $a_1, a_2, \dots, a_r$ are freely independent, selfadjoint operators in a C^*-probability space $(\mathcal{A}, \phi)$, then an inductive argument shows that all mixed moments

$$\phi(a_{i_1} a_{i_2} \cdots a_{i_p}), \qquad (p \in \mathbb{N},\ i_1, i_2, \dots, i_p \in \{1, 2, \dots, r\}),$$

are uniquely determined by the "marginals" $L\{a_1\}, L\{a_2\}, \dots, L\{a_r\}$. In particular, the moments (and hence the spectral distribution) of the sum $a_1 + a_2$ is uniquely determined by $L\{a_1\}$ and $L\{a_2\}$. This observation leads to the definition of free (additive) convolution:

2.3 Definition. Let μ_1 and μ_2 be compactly supported Borel probability measures on $\mathbb{R}$, and let a_1, a_2 be freely independent selfadjoint operators in a C^*-probability space $(\mathcal{A}, \phi)$, such that $L\{a_i\} = \mu_i$, $i = 1, 2$. Then the free additive convolution of μ_1 and μ_2 is the probability measure $\mu_1 \boxplus \mu_2$ on $\mathbb{R}$, given by: $\mu_1 \boxplus \mu_2 = L\{a_1 + a_2\}$.

The realization of freely independent, selfadjoint operators a_1 and a_2 with prescribed distributions μ_1 and μ_2 follows from a free product construction (see [VDN]). To extend the operation $\boxplus$ to all non compactly supported distributions on $\mathbb{R}$, one needs to take unbounded operators into account (see Subsection 2.3 below).

2.2 Free infinite divisibility and the Bercovici-Pata bijection.

Having introduced free (additive) convolution, we can define the corresponding notion of infinite divisibility in complete analogy with the classical case:

2.4 Definition. A probability measure μ on $\mathbb{R}$ is called $\boxplus$-infinitely divisible, if there exists, for each n in $\mathbb{N}$, a probability measure μ_n on $\mathbb{R}$, such that

$$\mu = \underbrace{\mu_n \boxplus \mu_n \boxplus \cdots \boxplus \mu_n}_{n \text{ terms}}.$$

We denote by $\mathcal{ID}(\boxplus)$ the class of $\boxplus$-infinitely divisible probability measures on $\mathbb{R}$.

In analogy with the definition above, we denote by $\mathcal{ID}(*)$ the class of infinitely divisible distributions w.r.t. classical convolution. As in classical probability, the $\boxplus$-infinitely divisible probability measures are characterised as those distributions for which the free cumulant transform has a Lévy-Khintchine type representation. Recall that a distribution μ is in $\mathcal{ID}(*)$ if and only if its (classical) cumulant transform C_μ (i.e., the logarithm of the characteristic function) has the Lévy-Khintchine representation:

$$C_\mu(u) = \mathrm{i}\eta u - \tfrac{1}{2} a u^2 + \int_{\mathbb{R}} \left(\mathrm{e}^{\mathrm{i}ut} - 1 - \mathrm{i}ut 1_{[-1,1]}(t)\right) \rho(\mathrm{d}t), \quad (u \in \mathbb{R}),$$

where $\eta \in \mathbb{R}$, $a \geq 0$ and ρ is a Lévy measure on $\mathbb{R}$, i.e.,

$$\rho(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} \min\{1, x^2\} \, \rho(\mathrm{d}x) < \infty.$$

The triplet (a, ρ, η) is uniquely determined, and it is called the generating triplet for μ.

In striking analogy, a probability measure μ on $\mathbb{R}$ belongs to $\mathcal{ID}(\boxplus)$ if and only if its free cumulant transform $\mathcal{C}_\mu$ has a representation in the form:

$$\mathcal{C}_\mu(z) = \eta z + az^2 + \int_{\mathbb{R}} \Big(\frac{1}{1-tz} - 1 - tz1_{[-1,1]}(t)\Big)\, \rho(\mathrm{d}t), \quad (z \in \mathbb{C},\ \mathrm{Im}(z) < 0).$$

In that case, the triplet (a, ρ, η) is, again, uniquely determined and is called the *free generating triplet* for μ.

The free cumulant transform $\mathcal{C}_\mu$ appearing above is a slight modification of the so-called R-transform, which was originally introduced by Voiculescu in [Vo1]. Its key property (proved, in the general case, in [BV]) is that it linearises free convolution, i.e., we have

$$\mathcal{C}_{\mu_1 \boxplus \mu_2}(z) = \mathcal{C}_{\mu_1}(z) + \mathcal{C}_{\mu_2}(z),$$

for any probability measures μ_1, μ_2 on $\mathbb{R}$.

With the Lévy-Khintchine representations (classical and free) established, it it clear that there is a bijection between $\mathcal{ID}(*)$ and $\mathcal{ID}(\boxplus)$. This bijection was first introduced and studied by Bercovici and Pata in [BP].

2.5 Definition. The Bercovici-Pata bijection is the mapping $\Lambda\colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$ defined in the following way: Suppose μ is in $\mathcal{ID}(*)$ and has generating triplet (a, ρ, η). Then $\Lambda(\mu)$ is the measure in $\mathcal{ID}(\boxplus)$ with free generating triplet (a, ρ, η).

At a first glance, the Bercovici-Pata bijection might seem as a very formal correspondence, but it was proved in [BP] that it maps the class of stable probability measures onto the class of freely stable probability measures. The following result further supports the significance of Λ:

2.6 Theorem. ([BT]) *The Bercovici-Pata bijection $\Lambda\colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$, satisfies:*

(i) If $\mu_1, \mu_2 \in \mathcal{ID}(*)$, then $\Lambda(\mu_1 * \mu_2) = \Lambda(\mu_1) \boxplus \Lambda(\mu_2)$.

(ii) If $\mu \in \mathcal{ID}(*)$ and $c \in \mathbb{R}$, then $\Lambda(D_c\mu) = D_c\Lambda(\mu)$.

(iii) For any c in $\mathbb{R}$, $\Lambda(\delta_c) = \delta_c$, where δ_c denotes the Dirac measure at c.

(iv) Λ is a homeomorphism w.r.t. weak convergence.

The operation D_c, appearing in (ii) of the above theorem, is dilation by the constant c. More precisely, if $\mu = L\{X\}$, for some random variable X, then $D_c\mu = L\{cX\}$. For proofs of the statements (i)-(iv) above, we refer to [BT].

2.7 Example. (a) Let μ be the standard Gaussian distribution, i.e.,

$$\mu(\mathrm{d}x) = \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}x^2)\,\mathrm{d}x.$$

Then $\Lambda(\mu)$ is the semi-circle distribution, i.e.,

$$\Lambda(\mu)(\mathrm{d}x) = \frac{1}{2\pi}\sqrt{4-x^2} \cdot 1_{[-2,2]}(x)\,\mathrm{d}x.$$

(b) Let μ be the classical Poisson distribution $\mathrm{Poiss}^*(\lambda)$ with mean $\lambda > 0$, i.e.,

$$\mu(\{n\}) = \mathrm{e}^{-\lambda}\frac{\lambda^n}{n!}, \qquad (n \in \mathbb{N}_0).$$

Then $\Lambda(\mu)$ is the free Poisson distribution $\mathrm{Poiss}^{\boxplus}(\lambda)$ with mean λ, i.e.,

$$\Lambda(\mu)(\mathrm{d}x) = \begin{cases} (1-\lambda)\delta_0 + \frac{1}{2\pi x}\sqrt{(x-a)(b-x)} \cdot 1_{[a,b]}(x)\,\mathrm{d}x, & \text{if } 0 \leq \lambda \leq 1, \\ \frac{1}{2\pi x}\sqrt{(x-a)(b-x)} \cdot 1_{[a,b]}(x)\,\mathrm{d}x, & \text{if } \lambda > 1, \end{cases}$$

where $a = (1-\sqrt{\lambda})^2$ and $b = (1+\sqrt{\lambda})^2$.

2.3 Unbounded operators.

As mentioned above, the (spectral) distribution of a selfadjoint operator in a C^*-probability space is a compactly supported probability measure. In order to realize a non compactly supported probability measure as the distribution of a selfadjoint operator, one needs, consequently, to take unbounded (i.e., non-continuous) operators into account.

Throughout this subsection, we consider a W^*-probability space $(\mathcal{A}, \tau)$, with $\mathcal{A}$ acting on the Hilbert space $\mathcal{H}$. Let $a\colon \mathcal{D}(a) \to \mathcal{H}$ be a (possibly unbounded) selfadjoint linear operator defined on a dense subspace $\mathcal{D}(a)$ of $\mathcal{H}$ (the "domain" of a). We say then that a is *affiliated with* $\mathcal{A}$, if $f(a) \in \mathcal{A}$ for any bounded Borel function $f\colon \mathbb{R} \to \mathbb{R}$ (here $f(a)$ is, again, defined in terms of functional calculus). In that case there exists, as in the bounded case, a unique probability measure μ_a on $(\mathbb{R}, \mathcal{B})$, such that

$$\int_{\mathbb{R}} f(t)\,\mu_a(\mathrm{d}t) = \tau(f(a)),$$

for any bounded Borel function $f\colon \mathbb{R} \to \mathbb{R}$. We refer to μ_a as the spectral distribution of a w.r.t. τ, and denote it also by $L\{a\}$. Given any probability measure μ on $\mathbb{R}$, it is not hard to see that μ can be realized as the spectral distribution of a selfadjoint operator affiliated with the von Neumann algebra $L^\infty(\mathbb{R}, \mu)$, namely the multiplication operator corresponding to the function $\mathrm{id}(t) = t$, $t \in \mathbb{R}$.

In general, a linear operator $a\colon \mathcal{D}(a) \to \mathcal{H}$, defined on a subspace $\mathcal{D}(a)$ of $\mathcal{H}$, is said to be affiliated with $\mathcal{A}$, if $au = ua$ for any unitary u in the commutant $\mathcal{A}'$ of $\mathcal{A}$. We say, furthermore, that a is *preclosed*, if the closure of its graph $\mathcal{G}(a) = \{(\xi, a\xi) \mid \xi \in \mathcal{D}(a)\}$ is, again, the graph of an operator $\overline{a}$. The operator $\overline{a}$ is then termed the closure of a. We say that a is closed, if $\mathcal{G}(a)$ is closed.

By $\overline{\mathcal{A}}$ we denote the set of closed, densely defined operators, which are affiliated with $\mathcal{A}$. If a and b are unbounded operators defined on subspaces $\mathcal{D}(a)$ and $\mathcal{D}(b)$ of $\mathcal{H}$, one needs, generally, to be cautious when performing operations like $a+b$ and ab, because one needs to keep track of the domains. If, however, a and b are both elements of $\overline{\mathcal{A}}$, then it can be shown (cf. e.g. [Ne]) that $a+b$ and ab are always densely defined, preclosed and with closures belonging to $\overline{\mathcal{A}}$. A similar statement holds for the adjoint operation. The closures of $a+b$ and ab are called, respectively, the strong sum and strong product of a and b. It can be shown, furthermore, that equipped with the strong sum, strong product and the adjoint operation, $\overline{\mathcal{A}}$ is a $*$-algebra (cf. [Se] or [Ne]). This fact allows us to work quite relaxed with the algebraic operations on unbounded operators in $\overline{\mathcal{A}}$, and we shall do so in the sequel.

2.8 Definition. Let $(\mathcal{A},\tau)$ be a W^*-probability space, with $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$.

(i) If $a_1, a_2, \dots, a_r$ are selfadjoint operators affiliated with $\mathcal{A}$, we say that they are *freely independent* w.r.t. τ, if

$$\tau\big\{[f_1(a_{i_1})-\tau(f_1(a_{i_1}))][f_2(a_{i_2})-\tau(f_2(a_{i_2}))]\cdots[f_p(a_{i_p})-\tau(f_p(a_{i_p}))]\big\}=0,$$

for any p in $\mathbb{N}$, any bounded Borel functions $f_1, f_2, \dots, f_p\colon \mathbb{R}\to\mathbb{R}$ and any indices $i_1, i_2, \dots, i_p$ in $\{1,2,\dots,r\}$ satisfying that $i_1\neq i_2, i_2\neq i_3, \dots, i_{p-1}\neq i_p$.

(ii) More generally, a family $S_1, S_2, \dots, S_r$ of sets of selfadjoint operators affiliated with $\mathcal{A}$ is called freely independent w.r.t. τ, if the following subsets of $\mathcal{A}$:

$$\{f(a)\mid a\in S_j,\ f\colon\mathbb{R}\to\mathbb{R},\ \text{bounded Borel function}\},\qquad (j=1,2,\dots,r),$$

are freely independent in the sense of Definition 2.2.

Finally, we need to introduce the notion of convergence in probability for operators in $\overline{\mathcal{A}}$.

2.9 Definition. Let $(\mathcal{A},\tau)$ be a W^*-probability space. A sequence (a_n) of operators in $\overline{\mathcal{A}}$ is said to converge *in probability* to a selfadjoint operator a in $\overline{\mathcal{A}}$, if

$$\forall\epsilon>0\colon \tau\big\{1_{]\epsilon,\infty[}(|a_n-a|)\big\}\longrightarrow 0,\qquad \text{as } n\to\infty.$$

In that case, we write $a_n \xrightarrow{\text{p}} a$, as $n\to\infty$.

It is not hard to see that $a_n\xrightarrow{\text{p}} a$, if and only if the spectral distributions $L\{|a_n-a|\}$ converge weakly to δ_0 (the Dirac measure at 0). If a_n and a are selfadjoint, this is again equivalent to asking that $L\{a_n-a\}\xrightarrow{\text{w}}\delta_0$, where $\xrightarrow{\text{w}}$ denotes weak convergence of probability measures.

2.10 Remark. The topology on $\overline{\mathcal{A}}$ corresponding to convergence in probability is the so-called *measure topology*. We note that all the algebraic operations in $\overline{\mathcal{A}}$ (i.e., scalar multiplication, strong sums and products and the adjoint operation) are continuous w.r.t. the measure topology (cf. [Se] or [Ne]). Furthermore, as we shall use repeatedly in the subsequent sections of this paper, the measure topology is a complete Hausdorff topology on $\overline{\mathcal{A}}$ (cf. [Ne]). In particular, if (a_n) is a sequence of selfadjoint operators affiliated with $\mathcal{A}$, such that

$$L\{a_n-a_m\}\xrightarrow{\text{w}}\delta_0,\quad \text{as } n,m\to\infty,$$

then there exists a unique selfadjoint operator a affiliated with $\mathcal{A}$, such that $a_n\xrightarrow{\text{p}} a$, as $n\to\infty$. We mention, finally, that convergence in the measure topology implies weak convergence of the corresponding (spectral) distributions (cf. [BT, Proposition 2.20]).

2.4 Lévy processes in free probability.

2.11 Definition. Let $(\mathcal{A},\tau)$ be a W^*-probability space. A free Lévy process (in law), affiliated with $(\mathcal{A},\tau)$, is a family $(Z_t)_{t\geq 0}$ of selfadjoint operators affiliated with $\mathcal{A}$, such that the following conditions are satisfied:

(i) whenever $n\in\mathbb{N}$ and $0\leq t_0<t_1<\cdots<t_n$, the increments

$$Z_{t_0}, Z_{t_1}-Z_{t_0}, Z_{t_2}-Z_{t_1},\dots,Z_{t_n}-Z_{t_{n-1}},$$

are freely independent selfadjoint operators affiliated with $\mathcal{A}$.

(ii) $Z_0=0$.

(iii) for any s,t in $[0,\infty[$, the (spectral) distribution of $Z_{s+t}-Z_s$ does not depend on s.

(iv) for any s in $[0,\infty[$, $Z_{s+t}\to Z_s$ in probability, as $t\to 0$, i.e. the (spectral) distributions $L\{Z_{s+t}-Z_s\}$ converge weakly to δ_0, as $t\to 0$.

The general existence of free Lévy processes (in law) has been noted in [Bi] and [Vo3] (cf. also Remark 6.7 below).

Recall that a classical Lévy process in law is a family $(X_t)_{t\geq 0}$ of classical random variables, defined on the same probability space $(\Omega,\mathcal{F},P)$, and satisfying conditions (i)-(iv) above, but with free independence replaced by classical independence in (i). If, in addition, the sample paths of (X_t) are cadlag with probability one, (X_t) is called a (genuine) Lévy process. Using the algebraic as well as the topological properties of Λ, it is not hard to derive the following one to one correspondence between classical and free Lévy processes (in law).

2.12 Proposition. ([BT]) *Let $(Z_t)_{t\geq 0}$ be a free Lévy process (in law) affiliated with a W^*-probability space $(\mathcal{A},\tau)$. Then there exists a (classical) Lévy process $(X_t)_{t\geq 0}$, defined on some probability space $(\Omega,\mathcal{F},P)$, such that $L\{X_t\}=\Lambda^{-1}(L\{Z_t\})$ for all t.*

Conversely, for any (classical) Lévy process (X_t), defined on a probability space $(\Omega,\mathcal{F},P)$, there exists a free Lévy process (in law), affiliated with some W^*-probability space $(\mathcal{A},\tau)$, such that $L\{Z_t\}=\Lambda(L\{X_t\})$ for all t.

2.13 Example. The *free Brownian motion* is the free Lévy process (in law), $(W_t)_{t\geq 0}$, which corresponds to the classical Brownian motion, $(B_t)_{t\geq 0}$, via the correspondence described in Proposition 2.12. In particular (cf. Example 2.7),

$$L\{W_t\}(\mathrm{d}s)=\frac{1}{2\pi t}\sqrt{4t-s^2}\cdot 1_{[-\sqrt{4t},\sqrt{4t}]}(s)\,\mathrm{d}s,\qquad (t\geq 0).$$

2.5 The classical Lévy-Itô decomposition.

As mentioned in the introduction, the Lévy-Itô decomposition represents a (classical) Lévy process (X_t) as the sum of two independent Lévy processes, the first of which is continuous (and hence a Brownian motion) and the second of which is, loosely speaking, the sum of the jumps of (X_t). In order to rigorously describe the sum of jumps part, one needs to introduce the notion of Poisson random measures. Before doing so, recall first that for any λ in $[0,\infty]$, we denote by $\text{Poiss}^*(\lambda)$ the (classical) Poisson distribution with mean λ. In particular, $\text{Poiss}^*(0) = \delta_0$ and $\text{Poiss}^*(\infty) = \delta_\infty$.

2.14 Definition. Let $(\Theta, \mathcal{E}, \nu)$ be a σ-finite measure space and let $(\Omega, \mathcal{F}, P)$ be a probability space. A Poisson random measure on $(\Theta, \mathcal{E}, \nu)$ and defined on $(\Omega, \mathcal{F}, P)$ is a mapping $N\colon \mathcal{E} \times \Omega \to [0,\infty]$, satisfying the following conditions:

(i) For each E in $\mathcal{E}$, $N(E) = N(E, \cdot)$ is a random variable on $(\Omega, \mathcal{F}, P)$.

(ii) For each E in $\mathcal{E}$, $L\{N(E)\} = \text{Poiss}^*(\nu(E))$.

(iii) If $E_1, \dots, E_n$ are disjoint sets from $\mathcal{E}$, then $N(E_1), \dots, N(E_n)$ are independent random variables.

(iv) For each fixed ω in Ω, the mapping $E \mapsto N(E, \omega)$ is a (positive) measure on $\mathcal{E}$.

In the setting of Definition 2.14, the measure ν is called the *intensity measure* for the Poisson random measure N. Let $(\Theta, \mathcal{E}, \nu)$ be a σ-finite measure space, and let N be a Poisson random measure on it (defined on some probability space $(\Omega, \mathcal{F}, P)$). Then for any $\mathcal{E}$-measurable function $f\colon \Theta \to [0,\infty]$, we may, for almost all ω in Ω, consider the integral $\int_\Theta f(\theta)\, N(\mathrm{d}\theta, \omega)$. We obtain, thus, an almost everywhere defined mapping on Ω, given by: $\omega \mapsto \int_\Theta f(\theta)\, N(\mathrm{d}\theta, \omega)$. This observation is the starting point for the theory of integration w.r.t. Poisson random measures, from which we shall need the following basic properties:

2.15 Proposition. *Let N be a Poisson random measure on the σ-finite measure space $(\Theta, \mathcal{E}, \nu)$, defined on the probability space $(\Omega, \mathcal{F}, P)$.*

(i) For any positive $\mathcal{E}$-measurable function $f\colon \Theta \to [0,\infty]$, $\int_\Theta f(\theta)\, N(\mathrm{d}\theta)$ is an $\mathcal{F}$-measurable positive function, and

$$\mathbb{E}\Big\{ \int_\Theta f(\theta)\, N(\mathrm{d}\theta) \Big\} = \int_\Theta f \,\mathrm{d}\nu.$$

(ii) If f is a real-valued function in $\mathcal{L}^1(\Theta, \mathcal{E}, \nu)$, then $f \in \mathcal{L}^1(\Theta, \mathcal{E}, N(\cdot, \omega))$ for almost all ω in Ω, $\int_\Theta f(\theta)\, N(\mathrm{d}\theta) \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$ and

$$\mathbb{E}\Big\{ \int_\Theta f(\theta)\, N(\mathrm{d}\theta) \Big\} = \int_\Theta f \,\mathrm{d}\nu.$$

The proof of the above proposition follows the usual pattern, proving it first for simple (positive) $\mathcal{E}$-measurable functions and then, via an approximation argument, obtaining the results in general. We shall adapt the same method in developing integration theory w.r.t. free Poisson random measures in Section 4 below.

We are now in a position to state the Lévy-Itô decomposition for classical Lévy processes. We denote the Lebesgue measure on $\mathbb{R}$ by Leb.

2.16 Theorem. (Lévy-Itô) *Let (X_t) be a classical (genuine) Lévy process and let ν be the Lévy measure appearing in the generating triplet for $L\{X_1\}$.*

(i) Assume that $\int_{-1}^{1} |x| \, \nu(\mathrm{d}x) < \infty$. Then (X_t) has a representation in the form:

$$X_t \overset{\text{a.s.}}{=} \gamma t + \sqrt{a} B_t + \int_{]0,t] \times \mathbb{R}} x \, N(\mathrm{d}s, \mathrm{d}x), \tag{2.1}$$

where $\gamma \in \mathbb{R}$, $a \geq 0$, (B_t) is a Brownian motion and N is a Poisson random measure on $(]0, \infty[\times \mathbb{R}, \mathrm{Leb} \otimes \nu)$. Furthermore, the last two terms on the right hand side of (2.1) are independent Lévy processes.

(ii) If $\int_{-1}^{1} |x| \, \nu(\mathrm{d}x) = \infty$, then we still have a decomposition like (2.1), but the integral $\int_{]0,t] \times \mathbb{R}} x \, N(\mathrm{d}s, \mathrm{d}x)$ no longer makes sense and has to be replaced by the limit:

$$Y_t = \lim_{\epsilon \searrow 0} \Big[\int_{]0,t] \times (\mathbb{R} \setminus [-\epsilon, \epsilon])} x \; N(\mathrm{d}u, \mathrm{d}x) - \int_{]0,t] \times ([-1,1] \setminus [-\epsilon, \epsilon])} x \, \mathrm{Leb} \otimes \nu(\mathrm{d}u, \mathrm{d}x) \Big], \quad (t \geq 0).$$

The process (Y_t) is, again, a Lévy process, which is independent of (B_t).

The symbol $\overset{\text{a.s.}}{=}$ in (2.1) means that the two random variables are equal with probability 1 (a.s. stands for "almost surely"). The Poisson random measure N appearing in the right hand side of (2.1) is, specifically, given by

$$N(E, \omega) = \#\big\{ s \in \,]0, \infty[\, \big| \; (s, \Delta X_s(\omega)) \in E \big\},$$

for any Borel subset E of $]0, \infty[\times (\mathbb{R} \setminus \{0\})$, and where $\Delta X_s = X_s - \lim_{u \nearrow s} X_u$. Consequently, the integral in the right hand side of (2.1) is, indeed, the sum of the jumps of X_t until time t: $\int_{]0,t] \times \mathbb{R}} x \; N(\mathrm{d}s, \mathrm{d}x) = \sum_{s \leq t} \Delta X_s$. The condition $\int_{-1}^{1} |x| \, \nu(\mathrm{d}x) < \infty$ ensures that this sum converges. Without that condition, one has to consider the "compensated sums of jumps" given by the process (Y_t). For a proof of Theorem 2.16 we refer to [Sa].

3 Free Poisson random measures and their existence.

In this section, we introduce free Poisson random measures and prove their existence. Recall, that for any number λ in $[0, \infty[$, we denote by $\mathrm{Poiss}^{\boxplus}(\lambda)$ the free Poisson distribution with mean λ (cf. Example 2.7).

3.1 Definition. Let $(\Theta, \mathcal{E}, \nu)$ be a measure space, and put

$$\mathcal{E}_0 = \{E \in \mathcal{E} \mid \nu(E) < \infty\}.$$

Let further $(\mathcal{A}, \tau)$ be a W^*-probability space, and let $\mathcal{A}_+$ denote the cone of positive operators in $\mathcal{A}$. Then a free Poisson random measure on $(\Theta, \mathcal{E}, \nu)$ with values in $(\mathcal{A}, \tau)$, is a mapping $M\colon \mathcal{E}_0 \to \mathcal{A}_+$, with the following properties:

(i) For any set E in $\mathcal{E}_0$, $L\{M(E)\} = \text{Poiss}^{\boxplus}(\nu(E))$.

(ii) If $r \in \mathbb{N}$ and $E_1, \dots, E_r$ are disjoint sets from $\mathcal{E}_0$, then $M(E_1), \dots, M(E_r)$ are freely independent operators.

(iii) If $r \in \mathbb{N}$ and $E_1, \dots, E_r$ are disjoint sets from $\mathcal{E}_0$, then $M(\cup_{j=1}^r E_j) = \sum_{j=1}^r M(E_j)$.

In the setting of Definition 3.1, the measure ν is called the *intensity measure* for the free Poisson random measure M. Note, in particular, that $M(E)$ is a *bounded* positive operator for all E in $\mathcal{E}_0$. The definition above might seem a little "poor" compared to that of a classical Poisson random measure. The following remark might offer a bit of consolation.

3.2 Remark. Suppose M is a free Poisson random measure on the measure space $(\Theta, \mathcal{E}, \nu)$ with values in the W^*-probability space $(\mathcal{A}, \tau)$. Let further (E_n) be a sequence of disjoint sets from $\mathcal{E}_0$. If we assume, in addition, that $\cup_{j\in\mathbb{N}} E_j \in \mathcal{E}_0$, then we also have that

$$M\Big(\bigcup_{j\in\mathbb{N}} E_j\Big) = \sum_{j=1}^{\infty} M(E_j),$$

where the right hand side should be understood as the limit *in probability* (cf. Subsection 2.3) of $\sum_{j=1}^n M(E_j)$ as $n \to \infty$.

Indeed, put $E = \cup_{j\in\mathbb{N}} E_j$, and assume that $E \in \mathcal{E}_0$. Then for any n in $\mathbb{N}$,

$$M(E) - \sum_{j=1}^{n} M(E_j) = M(E) - M(\cup_{j=1}^n E_j) = M(\cup_{j=n+1}^{\infty} E_j),$$

so that

$$L\Big\{M(E) - \sum_{j=1}^{n} M(E_j)\Big\} = \text{Poiss}^{\boxplus}\big(\nu(\cup_{j=n+1}^{\infty} E_j)\big) = \text{Poiss}^{\boxplus}\big(\textstyle\sum_{j=n+1}^{\infty} \nu(E_j)\big) \xrightarrow{\text{w}} \delta_0,$$

as $n \to \infty$, since $\sum_{j=n+1}^{\infty} \nu(E_j) \to 0$ as $n \to \infty$, because $\sum_{j=1}^{\infty} \nu(E_j) = \nu(E) < \infty$.

The main purpose of the section is to prove the general existence of free Poisson random measures.

3.3 Theorem. *Let $(\Theta, \mathcal{E}, \nu)$ be a measure space. Then there exists a W^*-probability space $(\mathcal{A}, \tau)$ and a free Poisson random measure M on $(\Theta, \mathcal{E}, \nu)$ with values in $(\mathcal{A}, \tau)$.*

The proof of Theorem 3.3 is given in a series of lemmas. First of all, though, we introduce some notation:

If $\mu_1, \mu_2, \ldots, \mu_r$ are probability measures on $\mathbb{R}$, we put:

$$\mathop{\boxplus}_{h=1}^{r} \mu_h = \mu_1 \boxplus \mu_2 \boxplus \cdots \boxplus \mu_r.$$

In the remaining part of this section, we consider the measure space $(\Theta, \mathcal{E}, \nu)$ appearing in Theorem 3.3. Consider then the set

$$\mathfrak{I} = \bigcup_{k\in\mathbb{N}} \{(E_1, \ldots, E_k) \mid E_1, \ldots, E_k \in \mathcal{E}_0 \setminus \{\emptyset\} \text{ and } E_1, \ldots, E_k \text{ are disjoint}\},$$

where we think of $(E_1, \ldots, E_k)$ merely as a collection of sets from $\mathcal{E}_0$. In particular, we identify $(E_1, \ldots, E_k)$ with $(E_{\pi(1)}, \ldots, E_{\pi(k)})$ for any permutation π of $\{1, 2, \ldots, k\}$. We introduce, furthermore, a partial order $\leq$ on $\mathfrak{I}$ by the convention:

$$(E_1, \ldots, E_k) \leq (F_1, \ldots, F_l) \iff \text{each } E_i \text{ is a union of some of the } F_j\text{'s}.$$

3.4 Lemma. *Given a tuple $S = (E_1, \ldots, E_k)$ from $\mathfrak{I}$, there exists a W^*-probability space $(\mathcal{A}_S, \tau_S)$, which is generated by freely independent positive operators $M_S(E_1), \ldots, M_S(E_k)$ from $\mathcal{A}_S$, satisfying that*

$$L\{M_S(E_i)\} = \mathrm{Poiss}^{\boxplus}(\nu(E_i)), \qquad (i = 1, \ldots, k).$$

Proof. This is an immediate consequence of Voiculescu's theory of (reduced) free products of von Neumann algebras (cf. [VDN]). Indeed, we may take $(\mathcal{A}_S, \tau_S)$ to be the (reduced) von Neumann algebra free product of the Abelian W^*-probability spaces $(L^\infty(\mathbb{R}, \mu_i), \mathbb{E}_{\mu_i})$, $i = 1, \ldots, k$, where $\mu_i = \mathrm{Poiss}^{\boxplus}(\nu(E_i))$ and $\mathbb{E}_{\mu_i}$ denotes expectation w.r.t. μ_i. ∎

3.5 Lemma. *Consider two elements $S = (E_1, \ldots, E_k)$ and $T = (F_1, \ldots, F_l)$ of $\mathfrak{I}$, and suppose that $S \leq T$. Consider the W^*-probability spaces $(\mathcal{A}_S, \tau_S)$ and $(\mathcal{A}_T, \tau_T)$ given by Lemma 3.4. Then there exists an injective, unital, normal $*$-homomorphism $\iota_{S,T} \colon \mathcal{A}_S \to \mathcal{A}_T$, such that $\tau_S = \tau_T \circ \iota_{S,T}$.*

Proof. We adapt the notation from Lemma 3.4. For any fixed i in $\{1, \ldots, k\}$, we have that $E_i = F_{j(i,1)} \cup \cdots \cup F_{j(i,l_i)}$, for suitable (distinct) $j(i,1), \ldots, j(i,l_i)$ from $\{1, 2, \ldots, l\}$. Note then that

$$\begin{aligned} L\big\{M_T(F_{j(i,1)}) + \cdots + M_T(F_{j(i,l_i)})\big\} &= \mathop{\boxplus}_{h=1}^{l_i} \mathrm{Poiss}^{\boxplus}(\nu(F_{j(i,h)})) \\ &= \mathrm{Poiss}^{\boxplus}\big(\nu(F_{j(i,1)}) + \cdots + \nu(F_{j(i,l_i)})\big) \\ &= \mathrm{Poiss}^{\boxplus}\big(\nu(F_{j(i,1)} \cup \cdots \cup F_{j(i,l_i)})\big) \\ &= \mathrm{Poiss}^{\boxplus}(\nu(E_i)) = L\{M_S(E_i)\}. \end{aligned}$$

In addition, $M_S(E_1), \ldots, M_S(E_k)$ are freely independent selfadjoint operators, and, similarly, the operators $\sum_{h=1}^{l_i} M_T(F_{j(i,h)})$, $i = 1, \ldots, k$ are freely independent and selfadjoint.

Combining these observations with [Vo2, Remark 1.8], it follows that there exists an injective, unital, normal $*$-homomorphism $\iota_{S,T}\colon \mathcal{A}_S \to \mathcal{A}_T$, such that

$$\iota_{S,T}(M_S(E_i)) = M_T(F_{j(i,1)}) + \cdots + M_T(F_{j(i,l_i)}), \qquad (i = 1, 2, \dots, r), \tag{3.1}$$

and such that $\tau_S = \tau_T \circ \iota_{S,T}$. ∎

3.6 Lemma. *Adapting the notation from Lemmas 3.4-3.5, the system*

$$(\mathcal{A}_S, \tau_S)_{S\in\mathfrak{I}}, \quad \{\iota_{S,T} \mid S, T \in \mathfrak{I},\ S \le T\}, \tag{3.2}$$

is a directed system of W^*-algebras and injective, unital, normal $*$-homomorphisms (cf. [KR, Section 11.4]).

Proof. Suppose that $R = (D_1, \dots, D_m)$, $S = (E_1, \dots, E_k)$ and $T = (F_1, \dots, F_l)$ are elements of $\mathfrak{I}$, such that $R \le S \le T$. We have to show that $\iota_{R,T} = \iota_{S,T} \circ \iota_{R,S}$. We may write (unambiguously),

$$\begin{aligned} D_h &= E_{i(h,1)} \cup \cdots \cup E_{i(h,k_h)}, \qquad (h = 1, \dots, m),\\ E_i &= F_{j(i,1)} \cup \cdots \cup E_{j(i,l_i)}, \qquad (i = 1, \dots, k), \end{aligned}$$

for suitable $i(h,1), \dots, i(h,k_h)$ in $\{1, 2, \dots, k\}$ and $j(i,1), \dots, j(i,l_i)$ in $\{1, 2, \dots, l\}$. Then for any h in $\{1, \dots, m\}$, we have

$$D_h = E_{i(h,1)} \cup \cdots \cup E_{i(h,k_h)} = \Big(\bigcup_{r=1}^{l_{i(h,1)}} F_{j(i(h,1),r)}\Big) \cup \cdots \cup \Big(\bigcup_{r=1}^{l_{i(h,k_h)}} F_{j(i(h,k_h),r)}\Big)$$

so that, by definition of $\iota_{R,T}$, $\iota_{R,S}$ and $\iota_{S,T}$ (cf. (3.1)),

$$\begin{aligned} \iota_{R,T}(D_h) &= \sum_{r=1}^{l_{i(h,1)}} M_T(F_{j(i(h,1),r)}) + \cdots + \sum_{r=1}^{l_{i(h,k_h)}} F_{j(i(h,k_h),r)}\\ &= \iota_{S,T}\big[M_S(E_{i(h,1)})\big] + \cdots + \iota_{S,T}\big[M_S(E_{i(h,k_h)})\big]\\ &= \iota_{S,T}\big[M_S(E_{i(h,1)}) + \cdots + M_S(E_{i(h,k_h)})\big]\\ &= \iota_{S,T}\big[\iota_{R,S}(D_h)\big]. \end{aligned}$$

Since $\mathcal{A}_R$ is generated, as a von Neumann algebra, by the operators $M_R(D_1), \dots, M_R(D_m)$, and since $\iota_{R,T}$ and $\iota_{S,T} \circ \iota_{R,S}$ are both normal $*$-homomorphisms, it follows by Kaplansky's density theorem (cf. [KR, Theorem 5.3.5]) and the calculation above that $\iota_{R,T} = \iota_{S,T} \circ \iota_{R,S}$, as desired. ∎

3.7 Lemma. *Let $\mathcal{A}^0$ denote the C^*-inductive limit of the directed system* (3.2) *and let $\iota_S\colon \mathcal{A}_S \to \mathcal{A}^0$ denote the canonical embedding of $\mathcal{A}_S$ into $\mathcal{A}^0$ (cf. [KR, Proposition 11.4.1]). Then there is a unique tracial state τ^0 on $\mathcal{A}^0$, satisfying that*

$$\tau_S = \tau^0 \circ \iota_S, \qquad \textit{for all } S \textit{ in } \mathfrak{I}. \tag{3.3}$$

Proof. Recall that the canonical embeddings $\iota_S\colon \mathcal{A}_S \to \mathcal{A}^0$ ($S \in \mathfrak{I}$) satisfy the condition:

$$\iota_R = \iota_S \circ \iota_{R,S}, \qquad \text{whenever } R, S \in \mathfrak{I} \text{ and } R \leq S.$$

We note first that (3.3) gives rise to a well-defined mapping τ^0 on the set $\mathcal{A}^{00} = \cup_{S\in\mathfrak{I}}\iota_S(\mathcal{A}_S)$. Indeed, suppose that $\iota_S(a') = \iota_T(a'')$ for some S, T in $\mathfrak{I}$ and $a' \in \mathcal{A}_S$, $a'' \in \mathcal{A}_T$. We need to show that $\tau_S(a') = \tau_T(a'')$. Let $S \vee T$ denote the tuple in $\mathfrak{I}$ consisting of all non-empty sets of the form $E \cap F$, where $E \in S$ and $F \in T$. Note that $S, T \leq S \vee T$. Since $\iota_S = \iota_{S\vee T} \circ \iota_{S,S\vee T}$ and $\iota_T = \iota_{S\vee T} \circ \iota_{T,S\vee T}$, it follows, by injectivity of $\iota_{S\vee T}$, that $\iota_{S,S\vee T}(a') = \iota_{T,S\vee T}(a'')$. Hence, by Lemma 3.5,

$$\tau_S(a') = \tau_{S\vee T} \circ \iota_{S,S\vee T}(a') = \tau_{S\vee T} \circ \iota_{T,S\vee T}(a'') = \tau_T(a''),$$

as desired. Now, given a, b in $\mathcal{A}^{00}$, we can find S from $\mathfrak{I}$, such that a, b are both in $\iota_S(\mathcal{A}_S)$, and hence it follows immediately that τ^0 is a linear tracial functional on the vector space $\mathcal{A}^{00}$. Furthermore, if $a = \iota_S(a')$ for some a' in $\mathcal{A}_S$, then

$$|\tau^0(a)| = |\tau_S(a')| \leq \|a'\| = \|\iota_S(a')\| = \|a\|,$$

so that τ^0 is norm decreasing. Since $\mathcal{A}^{00}$ is norm dense in $\mathcal{A}^0$ (cf. [KR, Proposition 11.4.1]), if follows then that τ^0 has a unique extension to a mapping $\tau^0\colon \mathcal{A}^0 \to \mathbb{C}$, which is automatically linear, tracial and norm-decreasing. In addition, $\tau^0(\mathbf{1}_{\mathcal{A}^0}) = 1 = \|\tau^0\|$, so, altogether, it follows that τ^0 is a tracial state on $\mathcal{A}^0$, satisfying (3.3). ∎

3.8 Lemma. *Let $(\mathcal{A}^0, \tau^0)$ be as in Lemma 3.7. There exists a mapping $M^0\colon \mathcal{E}_0 \to \mathcal{A}^0_+$, which satisfies conditions (i)-(iii) of Definition 3.1.*

Proof. We define M^0 by the equation:

$$M^0(E) = \iota_{\{E\}}(M_{\{E\}}(E)), \qquad (E \in \mathcal{E}_0).$$

Then $M^0(E)$ is positive for each E in $\mathcal{E}_0$, since $\iota_{\{E\}}$ is a $*$-homomorphism. Note also that if $E \in \mathcal{E}_0$ and $S \in \mathfrak{I}$ such that $E \in S$, then $\{E\} \leq S$ and

$$M^0(E) = \iota_{\{E\}}(M_{\{E\}}(E)) = \iota_S \circ \iota_{\{E\},S}(M_{\{E\}}(E)) = \iota_S(M_S(E)). \tag{3.4}$$

We now have

(i) For each E in $\mathcal{E}_0$, we have that $\tau_{\{E\}} = \tau^0 \circ \iota_{\{E\}}$, and hence, since $\iota_{\{E\}}$ is a $*$-homomorphism, $M_{\{E\}}(E)$ and $M^0(E)$ have the same moments w.r.t. $\tau_{\{E\}}$ and τ^0, respectively. Since both operators are bounded, this implies that $L\{M^0(E)\} = L\{M_{\{E\}}(E)\} = \text{Poiss}^{\boxplus}(\nu(E))$.

(ii) Let $E_1, \dots, E_k$ be disjoint sets from $\mathcal{E}_0$ and consider the tuple $S = (E_1, \dots, E_k) \in \mathfrak{I}$. Then, since $\tau_S = \tau^0 \circ \iota_S$ and ι_S is a $*$-homomorphism, we find, using (3.4),

$$\tau^0\big(M^0(E_{i_1})M^0(E_{i_2})\cdots M^0(E_{i_p})\big) = \tau_S\big(M_S(E_{i_1})M_S(E_{i_2})\cdots M_S(E_{i_p})\big),$$

for any $i_1, \dots, i_p$ in $\{1, 2, \dots, k\}$. Since $M_S(E_1), \dots, M_S(E_k)$ are freely independent, this implies that so are $M^0(E_1), \dots, M^0(E_k)$.

(iii) Let $E_1, \dots, E_k$ be disjoint sets from $\mathcal{E}_0$, put $E = \cup_{i=1}^k E_i$ and consider the tuple $S = (E_1, \dots, E_k) \in \mathcal{I}$. Then, by definition of $\iota_{\{E\},S}$, we have

$$\begin{aligned} M^0(E) &= \iota_{\{E\}}(M_{\{E\}}(E)) = \iota_S \circ \iota_{\{E\},S}(M_{\{E\}}(E)) = \iota_S\big(M_S(E_1) + \cdots + M_S(E_k)\big) \\ &= \iota_S(M_S(E_1)) + \cdots + \iota_S(M_S(E_k)) = M^0(E_1) + \cdots + M^0(E_k). \end{aligned}$$

This concludes the proof. ■

3.9 Lemma. *Let $(\mathcal{A}^0, \tau^0)$ be as in Lemma 3.7, let $\Phi^0 \colon \mathcal{A}^0 \to \mathcal{B}(\mathcal{H}^0)$ denote the GNS-representation*[1] *of $\mathcal{A}^0$ associated to τ^0, and let $\mathcal{A}$ be the closure of $\Phi^0(\mathcal{A}^0)$ in $\mathcal{B}(\mathcal{H}^0)$ w.r.t. the weak operator topology. Let, further, ξ^0 denote the unit vector in $\mathcal{H}^0$, which corresponds to the unit $\mathbf{1}_{\mathcal{A}^0}$ via the GNS-construction, and let τ denote the vector state on $\mathcal{A}$ given by ξ^0. Then $(\mathcal{A}, \tau)$ is a W^*-probability space, and $\tau^0 = \tau \circ \Phi^0$.*

Proof. It follows immediately from the GNS-construction that

$$\tau^0 = \tau \circ \Phi^0, \tag{3.5}$$

so we only have to prove that τ is a faithful trace on $\mathcal{A}$. To see that τ is a trace, note that since τ^0 is a trace, it follows from (3.5) that τ is a trace on the weakly dense C^*-subalgebra $\Phi^0(\mathcal{A}^0)$ of $\mathcal{A}$. Since the multiplication of operators is continuous w.r.t. the weak operator topology on bounded subsets of $\mathcal{B}(\mathcal{H}^0)$ and since τ is a vector state, it follows thus, by an application of Kaplansky's density theorem, that τ is a trace on all of $\mathcal{A}$. This means, furthermore, that ξ^0 is a generating trace vector for $\mathcal{A}$, and hence, by [KR, Lemma 7.2.14], it is also a generating trace vector for the commutant $\mathcal{A}' \subseteq \mathcal{B}(\mathcal{H}^0)$. This implies, in particular, that ξ^0 is separating for $\mathcal{A}$ (cf. [KR, Corollary 5.5.12]), which, in turn, implies that τ is faithful on $\mathcal{A}$. ■

Proof of Theorem 3.3. Let Φ^0 and $(\mathcal{A}, \tau)$ be as in Lemma 3.9. We then define the mapping $M \colon \mathcal{E}_0 \to \mathcal{A}_+$ by setting

$$M(E) = \Phi^0(M^0(E)), \qquad (E \in \mathcal{E}_0).$$

Now, Φ^0 is a $*$-homomorphism and $\tau^0 = \tau \circ \Phi^0$, so Φ^0 preserves all (mixed) moments of the elements $M^0(E)$, $E \in \mathcal{E}_0$. Since M^0 satisfies conditions (i)-(iii) of Definition 3.1, it follows thus, using the same line of argumentation as in the proof of Lemma 3.8, that M satisfies conditions (i)-(iii) too. Consequently, M is a free Poisson random measure on $(\Theta, \mathcal{E}, \nu)$ with values in $(\mathcal{A}, \tau)$. ■

4 Integration with respect to free Poisson random measures.

Throughout this section, we consider a free Poisson random measure M on the σ-finite measure space $(\Theta, \mathcal{E}, \nu)$ and with values in the W^*-probability space $(\mathcal{A}, \tau)$. We consider

[1] GNS stands for Gelfand-Naimark-Segal; see [KR, Theorem 4.5.2].

also a classical Poisson random measure N on $(\Theta, \mathcal{E}, \nu)$ defined on a classical probability space $(\Omega, \mathcal{F}, P)$. The aim of this section is to establish a theory of integration w.r.t. M, making sense, thus, to the integral $\int_\Theta f \, \mathrm{d}M$ for any function f in $\mathcal{L}^1(\Theta, \mathcal{E}, \nu)$. As in most theories of integration, we start by defining integration for simple ν-integrable functions.

4.1 Definition. Let s be a simple real-valued function in $\mathcal{L}^1(\Theta, \mathcal{E}, \nu)$, i.e., s can be written, unambiguously, in the form

$$s = \sum_{j=1}^{r} a_j 1_{E_j},$$

where $r \in \mathbb{N}$, $a_1, \ldots, a_r$ are *distinct* numbers in $\mathbb{R} \setminus \{0\}$ and $E_1, \ldots, E_r$ are *disjoint* sets from $\mathcal{E}_0$ (since s is ν-integrable). We then define the integral $\int_\Theta s \, \mathrm{d}M$ of s w.r.t. M as follows:

$$\int_\Theta s \, \mathrm{d}M = \sum_{j=1}^{r} a_j M(E_j) \in \mathcal{A}.$$

4.2 Remark. (a) Since $M(E) \in \mathcal{A}_+$ for any E in $\mathcal{E}_0$, it follows immediately from Definition 4.1 that $\int_\Theta s \, \mathrm{d}M$ is a selfadjoint operator in $\mathcal{A}$ for any simple real-valued function s in $\mathcal{L}^1(\Theta, \mathcal{E}, \mu)$.

(b) Suppose s is a simple real-valued function in $\mathcal{L}^1(\Theta, \mathcal{E}, \nu)$ written in the form: $s = \sum_{j=1}^{r} a_j 1_{E_j}$, where we assume that $r \in \mathbb{N}$, $a_1, \ldots, a_r \in \mathbb{R}$ and $E_1, \ldots, E_r \in \mathcal{E}_0$, but no longer that $a_1, \ldots, a_r$ are distinct or non-zero, nor that $E_1, \ldots, E_r$ are disjoint. Then, using standard arguments, it is not hard to see that we still have

$$\int_\Theta s \, \mathrm{d}M = \sum_{j=1}^{r} a_j M(E_j).$$

(c) Suppose s and t are simple real-valued functions in $\mathcal{L}^1(\Theta, \mathcal{E}, \nu)$ and that $c \in \mathbb{R}$. Then $s+t$ and $c \cdot s$ are clearly simple functions too, and, using (b) above, it is easily seen that

$$\int_\Theta (s+t) \, \mathrm{d}M = \int_\Theta s \, \mathrm{d}M + \int_\Theta t \, \mathrm{d}M, \quad \text{and} \quad \int_\Theta c \cdot s \, \mathrm{d}M = c \int_\Theta s \, \mathrm{d}M.$$

(d) Consider now, in addition, the classical Poisson random measure N on $(\Theta, \mathcal{E}, \nu)$, defined on $(\Omega, \mathcal{F}, P)$. Let, further, s be a simple real-valued function in $\mathcal{L}^1(\Theta, \mathcal{E}, \nu)$. Then $L\{\int_\Theta s \, \mathrm{d}N\} \in \mathcal{ID}(*)$, $L\{\int_\Theta s \, \mathrm{d}M\} \in \mathcal{ID}(\boxplus)$, and

$$\Lambda\Big(L\Big\{\int_\Theta s \, \mathrm{d}N\Big\}\Big) = L\Big\{\int_\Theta s \, \mathrm{d}M\Big\},$$

where Λ is the Bercovici-Pata bijection. Indeed, we may write s in the form $s = \sum_{j=1}^{r} a_j 1_{E_j}$, where $r \in \mathbb{N}$, $a_1, \ldots, a_r$ are distinct numbers in $\mathbb{R} \setminus \{0\}$ and $E_1, \ldots, E_r$

are disjoint sets from $\mathcal{E}_0$. Then, using the properties of Λ, we find that

$$L\Big\{\int_\Theta s\,\mathrm{d}M\Big\} = L\Big\{\sum_{j=1}^r a_j M(E_j)\Big\} = \boxplus_{j=1}^r D_{a_j}\mathrm{Poiss}^{\boxplus}(\nu(E_j))$$

$$= \boxplus_{j=1}^r D_{a_j}\Lambda\big[\mathrm{Poiss}^*(\nu(E_j))\big] = \Lambda\Big[\mathop{*}_{j=1}^r D_{a_j}\mathrm{Poiss}^*(\nu(E_j))\Big]$$

$$= \Lambda\Big[L\Big\{\sum_{j=1}^r a_j N(E_j)\Big\}\Big] = \Lambda\Big[L\Big\{\int_\Theta s\,\mathrm{d}N\Big\}\Big].$$

By $\mathcal{L}^1(\Theta,\mathcal{E},\nu)_+$, we denote the set of positive functions from $\mathcal{L}^1(\Theta,\mathcal{E},\nu)$.

4.3 Proposition. *Let f be a real-valued function in $\mathcal{L}^1(\Theta,\mathcal{E},\nu)$, and choose a sequence (s_n) of simple real-valued $\mathcal{E}$-measurable functions, satisfying the conditions:*

$$\exists h \in \mathcal{L}^1(\Theta,\mathcal{E},\nu)_+\ \forall \theta\in\Theta\ \forall n\in\mathbb{N}\colon\ |s_n(\theta)| \le h(\theta), \tag{4.1}$$

and

$$\lim_{n\to\infty} s_n(\theta) = f(\theta), \qquad (\theta\in\Theta). \tag{4.2}$$

Then $s_n \in \mathcal{L}^1(\Theta,\mathcal{E},\nu)$ for all n, and the integrals $\int_\Theta s_n\,\mathrm{d}M$ converge in probability to a selfadjoint (possibly unbounded) operator $I(f)$ affiliated with $\mathcal{A}$.

Furthermore, the limit $I(f)$ is independent of the choice of approximating sequence (s_n) of simple functions (subject to conditions (4.1) *and* (4.2)*).*

In condition (4.1), we might have taken $h = |f|$, but it is convenient to allow for more general dominators.

Proof of Proposition 4.3. Let f, (s_n) and h be as set out in the proposition. Then, for any n in $\mathbb{N}$, $\int_\Theta |s_n|\,\mathrm{d}\nu \le \int_\Theta h\,\mathrm{d}\nu < \infty$, so that $s_n \in \mathcal{L}^1(\Theta,\mathcal{E},\nu)$ and $\int_\Theta s_n\,\mathrm{d}M$ is well-defined. Note further that for any n,m in $\mathbb{N}$, $s_n - s_m$ is again a simple function in $\mathcal{L}^1(\Theta,\mathcal{E},\nu)$, and, using Remark 4.2(c),(d), it follows that

$$\begin{aligned} L\Big\{\int_\Theta s_n\,\mathrm{d}M - \int_\Theta s_m\,\mathrm{d}M\Big\} &= L\Big\{\int_\Theta (s_n - s_m)\,\mathrm{d}M\Big\} \\ &= \Lambda\Big[L\Big\{\int_\Theta (s_n - s_m)\,\mathrm{d}N\Big\}\Big], \end{aligned} \tag{4.3}$$

with N the classical Poisson random measure introduced before. Since $h \in \mathcal{L}^1(\Theta,\mathcal{E},\nu)$, it follows from Proposition 2.15 that $h \in \mathcal{L}^1(\Theta,\mathcal{E},M(\cdot,\omega))$ for almost all ω in Ω. Hence, by Lebesgue's theorem on dominated convergence, we have that

$$\int_\Theta s_n(\theta)\,N(\mathrm{d}\theta,\omega) \longrightarrow \int_\Theta f(\theta)\,N(\mathrm{d}\theta,\omega), \quad \text{as } n\to\infty,$$

for almost all ω in Ω. In other words, $\int_\Theta s_n \, \mathrm{d}N \to \int_\Theta f \, \mathrm{d}N$, almost surely, as $n \to \infty$. In particular $\int_\Theta s_n \, \mathrm{d}N \to \int_\Theta f \, \mathrm{d}N$, in probability as $n \to \infty$, so the sequence $(\int_\Theta s_n \, \mathrm{d}N)_{n \in \mathbb{N}}$ is a Cauchy sequence w.r.t. convergence in probability, i.e.,

$$L\Big\{ \int_\Theta (s_n - s_m) \, \mathrm{d}N \Big\} \xrightarrow{\text{w}} \delta_0, \quad \text{as } n, m \to \infty.$$

Combining this with (4.3) and the continuity of Λ (cf. Theorem 2.6), it follows that $(\int_\Theta s_n \, \mathrm{d}M)_{n \in \mathbb{N}}$ is also a Cauchy sequence w.r.t. convergence in probability, i.e., with respect to the measure topology. Since $\overline{\mathcal{A}}$ is complete w.r.t. the measure topology (cf. Remark 2.10), there exists, thus, an operator $I(f)$ in $\overline{\mathcal{A}}$, such that $\int_\Theta s_n \, \mathrm{d}M \to I(f)$, in probability as $n \to \infty$. Since $\int_\Theta s_n \, \mathrm{d}M$ is selfadjoint for each n, and since the adjoint operation is continuous w.r.t. the measure topology, $I(f)$ is a selfadjoint operator in $\overline{\mathcal{A}}$.

Suppose, finally, that (t_n) is another sequence of simple real-valued $\mathcal{E}$-measurable functions satisfying conditions (4.1) and (4.2) (with s_n replaced by t_n). Then, by the argument given above, $\int_\Theta t_n \, \mathrm{d}M \to I'(f)$, in probability as $n \to \infty$, for some selfadjoint operator $I'(f)$ in $\overline{\mathcal{A}}$. Consider now the mixed sequence (u_n) of simple real-valued $\mathcal{E}$-measurable functions given by:

$$u_1 = s_1, u_2 = t_1, u_3 = s_2, u_4 = t_2, \ldots,$$

and note that this sequence satisfies (4.1) and (4.2) too, so that $\int_\Theta u_n \, \mathrm{d}M \to I''(f)$, in probability as $n \to \infty$, for some selfadjoint operator $I''(f)$ in $\overline{\mathcal{A}}$. Now the subsequence (u_{2n-1}) converges in probability to both $I''(f)$ and $I(f)$ as $n \to \infty$, and the subsequence (u_{2n}) converges in probability to both $I''(f)$ and $I'(f)$ as $n \to \infty$. Since the measure topology is a Hausdorff topology, we may conclude, thus, that $I(f) = I''(f) = I'(f)$. This completes the proof. ■

4.4 Definition. Let f be a real-valued function in $\mathcal{L}^1(\Theta, \mathcal{E}, \nu)$, and let $I(f)$ be the selfadjoint operator in $\overline{\mathcal{A}}$ described in Proposition 4.3. We call $I(f)$ the integral of f w.r.t. M and denote it by $\int_\Theta f \, \mathrm{d}M$.

4.5 Corollary. *Let M and N be the free and classical Poisson random measures on $(\Theta, \mathcal{E}, \nu)$ introduced above. Then for any f in $\mathcal{L}^1(\Theta, \mathcal{E}, \nu)$, we have $L\{\int_\Theta f \, \mathrm{d}N\} \in \mathcal{ID}(*)$, $L\{\int_\Theta f \, \mathrm{d}M\} \in \mathcal{ID}(\boxplus)$ and*

$$\Lambda\Big(L\Big\{ \int_\Theta f \, \mathrm{d}N \Big\}\Big) = L\Big\{ \int_\Theta f \, \mathrm{d}M \Big\}.$$

Proof. Choose a sequence (s_n) of simple real-valued $\mathcal{E}$-measurable functions satisfying conditions (4.1) and (4.2) of Proposition 4.3. Then, by Remark 4.2, $L\{\int_\Theta s_n \, \mathrm{d}N\} \in \mathcal{ID}(*)$, $L\{\int_\Theta s_n \, \mathrm{d}M\} \in \mathcal{ID}(\boxplus)$ and $\Lambda(L\{\int_\Theta s_n \, \mathrm{d}N\}) = L\{\int_\Theta s_n \, \mathrm{d}M\}$ for all n in $\mathbb{N}$. Furthermore,

$$\int_\Theta s_n \, \mathrm{d}N \xrightarrow{\text{a.s.}} \int_\Theta f \, \mathrm{d}N \quad \text{and} \quad \int_\Theta s_n \, \mathrm{d}M \xrightarrow{\text{P}} \int_\Theta f \, \mathrm{d}M, \quad \text{as } n \to \infty.$$

In particular (cf. Remark 2.10),

$$L\Big\{ \int_\Theta s_n \, \mathrm{d}N \Big\} \xrightarrow{\text{w}} L\Big\{ \int_\Theta f \, \mathrm{d}N \Big\} \quad \text{and} \quad L\Big\{ \int_\Theta s_n \, \mathrm{d}M \Big\} \xrightarrow{\text{w}} L\Big\{ \int_\Theta f \, \mathrm{d}M \Big\}, \quad \text{as } n \to \infty.$$

Since $\mathcal{ID}(*)$ and $\mathcal{ID}(\boxplus)$ are both closed w.r.t. weak convergence (see [Pa]), this implies that $L\{\int_\Theta f \,\mathrm{d}N\} \in \mathcal{ID}(*)$ and $L\{\int_\Theta f \,\mathrm{d}M\} \in \mathcal{ID}(\boxplus)$. Furthermore, by continuity of Λ, $\Lambda(L\{\int_\Theta f \,\mathrm{d}N\}) = L\{\int_\Theta f \,\mathrm{d}M\}$. ■

4.6 Proposition. *For any real-valued functions f, g in $\mathcal{L}^1(\Theta, \mathcal{E}, \nu)$ and any real number c, we have that*

$$\int_\Theta (f+g)\,\mathrm{d}M = \int_\Theta f\,\mathrm{d}M + \int_\Theta g\,\mathrm{d}M \quad \text{and} \quad \int_\Theta c\cdot f\,\mathrm{d}M = c\int_\Theta f\,\mathrm{d}M.$$

Proof. If f and g are simple functions, this was noted in Remark 4.2. The general case follows by approximating f and g by simple functions as in Proposition 4.3 and using that addition and scalar-multiplication are continuous operations w.r.t. the measure topology (cf. Remark 2.10). ■

5 Free independence and convergence in probability.

In this section, we study the relationship between convergence in probability and free independence. The results will be used in the proof of the free Lévy-Itô decomposition in Section 6 below.

5.1 Lemma. *Let (b_n) be a sequence of (not necessarily selfadjoint) operators in a W^*-probability space $(\mathcal{A}, \tau)$, and assume that $\|b_n\| \le 1$ for all n. Assume, further, that $b_n \to b$ in probability as $n \to \infty$ for some operator b in $\mathcal{A}$, such that $\|b\| \le 1$. Then also $\tau(b_n) \to \tau(b)$, as $n \to \infty$.*

Proof. For each n in $\mathbb{N}$, put $b_n' = \frac{1}{2}(b + b^*)$ and $b_n'' = \frac{1}{2\mathrm{i}}(b_n - b_n^*)$, and define b', b'' similarly from b. Then b_n', b_n'', b', b'' are all selfadjoint operators in $\mathcal{A}$ of norm less than or equal to 1. Since addition, scalar-multiplication and the adjoint operation are all continuous operations w.r.t. the measure topology, it follows, furthermore, that $b_n' \to b'$ and $b_n'' \to b''$ in probability as $n \to \infty$. This implies that $b_n' \to b_n'$ and $b_n'' \to b''$ in distribution as $n \to \infty$, i.e., that $L\{b_n'\} \xrightarrow{\mathrm{w}} L\{b'\}$ and $L\{b_n''\} \xrightarrow{\mathrm{w}} L\{b''\}$ as $n \to \infty$ (cf. Remark 2.10).

Now, choose a continuous bounded function $f\colon \mathbb{R} \to \mathbb{R}$, such that $f(x) = x$ for all x in $[-1, 1]$. Then, since $\mathrm{sp}(b_n'), \mathrm{sp}(b')$ are contained in $[-1, 1]$, we find that

$$\tau(b_n') = \tau(f(b_n')) = \int_{\mathbb{R}} f(x)\, L\{b_n'\}(\mathrm{d}x) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}} f(x)\, L\{b'\}(\mathrm{d}x) = \tau(f(b')) = \tau(b').$$

Similarly, $\tau(b_n'') \to \tau(b'')$ as $n \to \infty$, and hence also $\tau(b_n) = \tau(b_n' + \mathrm{i}b_n'') \to \tau(b' + \mathrm{i}b'') = \tau(b)$, as $n \to \infty$. ■

5.2 Lemma. *Let r be a positive integer, and let $(b_{1,n})_{n\in\mathbb{N}}, \dots, (b_{r,n})_{n\in\mathbb{N}}$ be sequences of bounded (not necessarily selfadjoint) operators in the W^*-probability space $(\mathcal{A}, \tau)$. Assume, for each j, that $\|b_{j,n}\| \le 1$ for all n and that $b_{j,n} \to b_j$ in probability as $n \to \infty$, for some operator b_j in $\mathcal{A}$, such that $\|b_j\| \le 1$. If $b_{1,n}, b_{2,n}, \dots, b_{r,n}$ are freely independent for each n, then the operators $b_1, b_2, \dots, b_r$ are also freely independent.*

Proof. Assume that $b_{1,n}, b_{2,n}, \dots, b_{r,n}$ are freely independent for all n, and let $i_1, i_2, \dots, i_p$ in $\{1, 2, \dots, r\}$ be given. Then there is a universal polynomial $P_{i_1,\dots,i_p}$ in rp complex variables, depending only on $i_1, \dots, i_p$, such that for all n in $\mathbb{N}$,

$$\tau(b_{i_1,n} b_{i_2,n} \cdots b_{i_p,n}) = P_{i_1,\dots,i_p}\Big[\big\{\tau(b_{1,n}^\ell)\big\}_{1\le\ell\le p}, \dots, \big\{\tau(b_{r,n}^\ell)\big\}_{1\le\ell\le p}\Big]. \tag{5.1}$$

Now, since operator multiplication is a continuous operation w.r.t. the measure topology, $b_{i_1,n} b_{i_2,n} \cdots b_{i_p,n} \to b_{i_1} b_{i_2} \cdots b_{i_p}$ in probability as $n \to \infty$. Furthermore, $\|b_{i_1,n} b_{i_2,n} \cdots b_{i_p,n}\| \le 1$ for all n and $\|b_{i_1} b_{i_2} \cdots b_{i_p}\| \le 1$, so by Lemma 5.1 we have

$$\tau\big(b_{i_1,n} b_{i_2,n} \cdots b_{i_p,n}\big) \underset{n\to\infty}{\longrightarrow} \tau\big(b_{i_1} b_{i_2} \cdots b_{i_p}\big).$$

Similarly,

$$\tau(b_{j,n}^\ell) \underset{n\to\infty}{\longrightarrow} \tau(b_j^\ell), \quad \text{for any } j \text{ in } \{1, 2, \dots, r\} \text{ and } \ell \text{ in } \mathbb{N}.$$

Combining these observations with (5.1), we conclude that also

$$\tau(b_{i_1} b_{i_2} \cdots b_{i_p}) = P_{i_1,\dots,i_p}\Big[\big\{\tau(b_1^\ell)\big\}_{1\le\ell\le p}, \dots, \big\{\tau(b_r^\ell)\big\}_{1\le\ell\le p}\Big],$$

and since this holds for arbitrary $i_1, \dots, i_p$ in $\{1, 2, \dots, r\}$, it follows that $b_1, \dots, b_r$ are freely independent, as desired. ■

For a selfadjoint operator a affiliated with a W^*-probability space $(\mathcal{A}, \tau)$, we denote by $\kappa(a)$ the *Cayley transform* of a, i.e.,

$$\kappa(a) = (a - \mathrm{i}\mathbf{1}_{\mathcal{A}})(a + \mathrm{i}\mathbf{1}_{\mathcal{A}})^{-1}.$$

Recall that even though a may be an unbounded operator, $\kappa(a)$ is a unitary operator in $\mathcal{A}$.

5.3 Lemma. *Let $a_1, a_2, \dots, a_r$ be selfadjoint operators affiliated with the W^*-probability space $(\mathcal{A}, \tau)$. Then $a_1, a_2, \dots, a_r$ are freely independent if and only if $\kappa(a_1), \kappa(a_2), \dots, \kappa(a_r)$ are freely independent.*

Proof. For each j in $\{1, 2, \dots, r\}$, let $W^*\{a_j\}$ denote the von Neumann subalgebra of $\mathcal{A}$ generated by a_j, i.e.,

$$W^*\{a_j\} = \{f(a_j) \mid f\colon \mathbb{R} \to \mathbb{C} \text{ is a bounded Borel function}\}.$$

Similarly, let $W^*\{\kappa(a_j)\}$ denote the von Neumann subalgebra of $\mathcal{A}$ generated by the unitary $\kappa(a_j)$. Now, $a_1, \dots, a_r$ are freely independent if and only if the subalgebras $W^*\{a_1\}, \dots, W^*\{a_r\}$ are freely independent, and, similarly, $\kappa(a_1), \dots, \kappa(a_r)$ are freely independent if and only if the subalgebras $W^*\{\kappa(a_1)\}, \dots, W^*\{\kappa(a_r)\}$ are freely independent. The lemma follows then by recalling that $W^*\{a_j\} = W^*\{\kappa(a_j)\}$ for all j (cf. [Pe, Lemma 5.2.8]). ■

5.4 Proposition. *Suppose $r \in \mathbb{N}$ and that $(a_{1,n})_{n\in\mathbb{N}}, \dots, (a_{r,n})_{n\in\mathbb{N}}$ are sequences of selfadjoint operators affiliated with the W^*-probability space $(\mathcal{A}, \tau)$. Assume, further, that for each j in $\{1, 2, \dots, r\}$, $a_{j,n} \to a_j$ in probability as $n \to \infty$, for some selfadjoint operator a_j affiliated with $(\mathcal{A}, \tau)$. If $a_{1,n}, a_{2,n}, \dots, a_{r,n}$ are freely independent for each n, then the operators $a_1, a_2, \dots, a_r$ are also freely independent.*

Proof. Assume that $a_{1,n}, a_{2,n}, \dots, a_{r,n}$ are freely independent for all n. Then, by Lemma 5.3, the unitaries $\kappa(a_{1,n}), \dots, \kappa(a_{r,n})$ are freely independent for each n in $\mathbb{N}$. Moreover, since the Cayley transform is continuous w.r.t. the measure topology (cf. [St, Lemma 5.3]), we have

$$\kappa(a_{j,n}) \underset{n\to\infty}{\longrightarrow} \kappa(a_j), \qquad \text{in probability,}$$

for each j. Hence, by Lemma 5.2, the unitaries $\kappa(a_1), \dots, \kappa(a_r)$ are freely independent, and, appealing once more to Lemma 5.3, this means that $a_1, \dots, a_r$ themselves are freely independent. ∎

5.5 Corollary. *Let M be a free Poisson random measure on the σ-finite measure space $(\Theta, \mathcal{E}, \nu)$ with values in the W^*-probability space $(\mathcal{A}, \tau)$. Let, further, $f_1, f_2, \dots, f_r$ be real-valued functions in $\mathcal{L}^1(\Theta, \mathcal{E}, \nu)$ and let $\Theta_1, \Theta_2, \dots, \Theta_r$ be disjoint $\mathcal{E}$-measurable subsets of Θ. Then the integrals*

$$\int_{\Theta_1} f_1 \, \mathrm{d}M, \int_{\Theta_2} f_2 \, \mathrm{d}M, \dots, \int_{\Theta_r} f_r \, \mathrm{d}M,$$

are freely independent selfadjoint operators affiliated with $(\mathcal{A}, \tau)$.

Proof. For each j in $\{1, 2, \dots, r\}$, let $(s_{j,n})_{n\in\mathbb{N}}$ be a sequence of real valued simple $\mathcal{E}$-measurable functions, such that

$$|s_{j,n}(\theta)| \le |f_j(\theta)|, \qquad (\theta \in \Theta, \ n \in \mathbb{N}),$$

and

$$\lim_{n\to\infty} s_{j,n}(\theta) = f_j(\theta), \qquad (\theta \in \Theta).$$

Then, for each j in $\{1, 2, \dots, r\}$ and each n in $\mathbb{N}$, we may write $s_{j,n} \cdot 1_{\Theta_j}$ in the form:

$$s_{j,n} \cdot 1_{\Theta_j} = \sum_{l=1}^{k_{j,n}} \alpha(l, j, n) 1_{A(l,j,n)},$$

where $\alpha(1, j, n), \dots, \alpha(k_{j,n}, j, n) \in \mathbb{R} \setminus \{0\}$ and $A(1, j, n), \dots, A(k_{j,n}, j, n)$ are disjoint sets from $\mathcal{E}_0$, such that $A(l, j, n) \subseteq \Theta_j$ for all l. Now,

$$\int_{\Theta} s_{j,n} \cdot 1_{\Theta_j} \, \mathrm{d}M = \sum_{l=1}^{k_{j,n}} \alpha(l, j, n) M((A(l, j, n)), \qquad (j = 1, 2, \dots, r, \ n \in \mathbb{N}),$$

so by the properties of free Poisson random measures, the integrals

$$\int_{\Theta} s_{1,n} \cdot 1_{\Theta_1} \, \mathrm{d}M, \dots, \int_{\Theta} s_{r,n} \cdot 1_{\Theta_r} \, \mathrm{d}M,$$

are freely independent for each n in $\mathbb{N}$. Finally, for each j in $\{1,2,\dots,r\}$ we have (cf. Proposition 4.3)

$$\int_{\Theta_j} f_j \,\mathrm{d}M = \int_{\Theta} f_j \cdot 1_{\Theta_j} \,\mathrm{d}M = \lim_{n\to\infty} \int_{\Theta} s_{j,n} \cdot 1_{\Theta_j} \,\mathrm{d}M,$$

where the limit is taken in probability. Taking now Proposition 5.4 into account, we obtain the desired conclusion. ∎

5.6 Remark. Let $\mathcal{B}$ and $\mathcal{C}$ be two freely independent von Neumann subalgebras of a W^*-probability space $(\mathcal{A},\tau)$. Let, further, (b_n) and (c_n) be two sequences of selfadjoint operators, which are affiliated with $\mathcal{B}$ and $\mathcal{C}$, respectively, in the sense that $f(b_n) \in \mathcal{B}$ and $g(c_n) \in \mathcal{C}$ for any n in $\mathbb{N}$ and any bounded Borel functions $f, g\colon \mathbb{R} \to \mathbb{R}$. Assume that $b_n \to b$ and $c_n \to c$ in probability as $n \to \infty$. Then b and c are also freely independent. This follows, of course, from Proposition 5.4, but it is also an immediate consequence of the fact that the set $\overline{\mathcal{B}}$ of closed, densely defined operators, affiliated with $\mathcal{B}$, is complete (and hence closed) w.r.t. the measure topology. Indeed, the restriction to $\overline{\mathcal{B}}$ of the measure topology on $\overline{\mathcal{A}}$ is the measure topology on $\overline{\mathcal{B}}$ (induced by $\tau_{|\mathcal{B}}$). Thus, b is affiliated with $\mathcal{B}$ and similarly c is affiliated with $\mathcal{C}$, so that, in particular, b and c are freely independent.

6 The Free Lévy-Itô Decomposition.

Throughout this section we put

$$\mathsf{H} = \,]0,\infty[\times\mathbb{R} \subseteq \mathbb{R}^2,$$

and we denote by $\mathsf{B}(\mathsf{H})$ the set of all Borel subsets of H. Furthermore, for any ϵ, t in $]0,\infty[$, we put

$$D(\epsilon,\infty) = \{s \in \mathbb{R} \mid \epsilon < |s| < \infty\} = \mathbb{R} \setminus [-\epsilon,\epsilon],$$
$$D(\epsilon,t) = \{s \in \mathbb{R} \mid \epsilon < |s| \le t\} = [-t,t] \setminus [-\epsilon,\epsilon].$$

We shall need the following well-known result about classical Poisson random measures.

6.1 Lemma. *Let ν be a Lévy measure on $\mathbb{R}$ and consider the σ-finite measure* $\mathrm{Leb}\otimes\nu$ *on H. Consider further a (classical) Poisson random measure N on $(\mathsf{H},\mathsf{B}(\mathsf{H}),\mathrm{Leb}\otimes\nu)$, defined on some probability space $(\Omega,\mathcal{F},P)$.*

Then there is a subset Ω_0 of Ω, such that $\Omega_0 \in \mathcal{F}$, $P(\Omega_0) = 1$ and such that the following holds for any ω in Ω_0: For any ϵ, t in $]0,\infty[$, the restriction $[N(\cdot,\omega)]_{]0,t]\times D(\epsilon,\infty)}$ of the measure $N(\cdot,\omega)$ to the set $]0,t] \times D(\epsilon,\infty)$ is supported on a finite number of points, each of which has mass 1.

Proof. See [Sa, Lemma 20.1] ∎

6.2 Lemma. *Let ν and N be as in Lemma 6.1, and consider a positive Borel function $\varphi\colon \mathbb{R} \to [0, \infty[$.*

(i) For almost all ω in Ω, the following holds:

$$\forall \epsilon > 0\ \forall 0 \leq s < t\colon \int_{]s,t]\times D(\epsilon,\infty)} \varphi(x)\, N(\mathrm{d}u, \mathrm{d}x, \omega) < \infty.$$

(ii) If $\int_{[-1,1]} \varphi(x)\, \nu(\mathrm{d}x) < \infty$, then for almost all ω in Ω, the following holds:

$$\forall 0 \leq s < t\colon \int_{]s,t]\times\mathbb{R}} \varphi(x)\, N(\mathrm{d}u, \mathrm{d}x, \omega) < \infty.$$

Proof. Since φ is positive, it suffices to consider the case $s = 0$ in (i) and (ii). Moreover, since φ only takes finite values, statement (i) follows immediately from Lemma 6.1.

To prove (ii), assume that $\int_{[-1,1]} \varphi(x)\, \nu(\mathrm{d}x) < \infty$. By virtue of (i), it suffices then to prove, for instance, that for almost all ω in Ω, the following holds:

$$\forall t > 0\colon \int_{]0,t]\times[-1,1]} \varphi(x)\, N(\mathrm{d}u, \mathrm{d}x, \omega) < \infty. \tag{6.1}$$

Since the integrals in (6.1) increase with t, it suffices to prove that for any fixed t in $]0, \infty[$,

$$\int_{]0,t]\times[-1,1]} \varphi(x)\, N(\mathrm{d}u, \mathrm{d}x, \omega) < \infty, \quad \text{for almost all } \omega.$$

This, in turn, follows immediately from the following calculation:

$$\begin{aligned}\mathbb{E}\Big\{ \int_{]0,t]\times[-1,1]} \varphi(x)\, N(\mathrm{d}u, \mathrm{d}x) \Big\} &= \int_{]0,t]\times[-1,1]} \varphi(x)\, \mathrm{Leb}\otimes\nu(\mathrm{d}u, \mathrm{d}x) \\ &= t \int_{[-1,1]} \varphi(x)\, \nu(\mathrm{d}x) < \infty,\end{aligned}$$

where we have used Proposition 2.15. ■

6.3 Lemma. *Let ν be a Lévy measure on $\mathbb{R}$, and let M be a Free Poisson random measure on $(\mathsf{H}, \mathsf{B}(\mathsf{H}), \mathrm{Leb}\otimes\nu)$ with values in the W^*-probability space $(\mathcal{A}, \tau)$. Let, further, N be a (classical) Poisson random measure on $(\mathsf{H}, \mathsf{B}(\mathsf{H}), \mathrm{Leb}\otimes\nu)$, defined on a classical probability space $(\Omega, \mathcal{F}, P)$.*

(i) For any ϵ, s, t in $]0, \infty[$, such that $s < t$, the integrals

$$\int_{]s,t]\times D(\epsilon,n)} x\, M(\mathrm{d}u, \mathrm{d}x), \qquad (n \in \mathbb{N}),$$

converge in probability, as $n \to \infty$, to some (possibly unbounded) selfadjoint operator affiliated with $\mathcal{A}$, which we denote by $\int_{]s,t]\times D(\epsilon,\infty)} x\, M(\mathrm{d}u, \mathrm{d}x)$. Furthermore *(cf. Lemma 6.2),* $\int_{]s,t]\times D(\epsilon,\infty)} x\, N(\mathrm{d}u,\mathrm{d}x) \in \mathcal{ID}(*)$*,* $\int_{]s,t]\times D(\epsilon,\infty)} x\, M(\mathrm{d}u,\mathrm{d}x) \in \mathcal{ID}(\boxplus)$ *and*

$$L\Big\{\int_{]s,t]\times D(\epsilon,\infty)} x\, M(\mathrm{d}u,\mathrm{d}x)\Big\} = \Lambda\Big(L\Big\{\int_{]s,t]\times D(\epsilon,\infty)} x\, N(\mathrm{d}s,\mathrm{d}x)\Big\}\Big). \tag{6.2}$$

(ii) If $\int_{[-1,1]} |x|\, \nu(\mathrm{d}x) < \infty$*, then for any* s,t *in* $]0,\infty[$*, such that* $s<t$*, the integrals*

$$\int_{]s,t]\times[-n,n]} x\, M(\mathrm{d}u,\mathrm{d}x), \qquad (n\in\mathbb{N}),$$

converge in probability, as $n\to\infty$*, to some (possibly unbounded) selfadjoint operator affiliated with* $\mathcal{A}$*, which we denote by* $\int_{]s,t]\times\mathbb{R}} x\, M(\mathrm{d}u,\mathrm{d}x)$*. Furthermore (cf. Lemma 6.2),* $\int_{]s,t]\times\mathbb{R}} x\, N(\mathrm{d}u,\mathrm{d}x) \in \mathcal{ID}(*)$*,* $\int_{]s,t]\times\mathbb{R}} x\, M(\mathrm{d}u,\mathrm{d}x) \in \mathcal{ID}(\boxplus)$ *and*

$$L\Big\{\int_{]s,t]\times\mathbb{R}} x\, M(\mathrm{d}u,\mathrm{d}x)\Big\} = \Lambda\Big(L\Big\{\int_{]s,t]\times\mathbb{R}} x\, N(\mathrm{d}s,\mathrm{d}x)\Big\}\Big).$$

Proof. (i) Note first that for any n in $\mathbb{N}$ and any ϵ,s,t in $]0,\infty[$, such that $s<t$, we have that

$$\int_{]s,t]\times D(\epsilon,n)} |x|\, \mathrm{Leb}\otimes\nu(\mathrm{d}u,\mathrm{d}x) = (t-s)\int_{D(\epsilon,n)} |x|\,\nu(\mathrm{d}x) < \infty,$$

since ν is a Lévy measure. Hence, by Proposition 4.3, the integral $\int_{]s,t]\times D(\epsilon,n)} x\, M(\mathrm{d}u,\mathrm{d}x)$ is well-defined and furthermore, by Corollary 4.5,

$$L\Big\{\int_{]s,t]\times D(\epsilon,n)} x\, M(\mathrm{d}u,\mathrm{d}x)\Big\} = \Lambda\Big(L\Big\{\int_{]s,t]\times D(\epsilon,n)} x\, N(\mathrm{d}u,\mathrm{d}x)\Big\}\Big). \tag{6.3}$$

Note now that by Lemma 6.2(i) there is a subset Ω_0 of Ω, such that $\Omega_0\in\mathcal{F}$, $P(\Omega_0)=1$ and

$$\int_{]s,t]\times D(\epsilon,\infty)} |x|\, N(\mathrm{d}u,\mathrm{d}x,\omega) < \infty, \quad \text{for all } \omega \text{ in } \Omega_0.$$

Then $\int_{]s,t]\times D(\epsilon,\infty)} x\, N(\mathrm{d}u,\mathrm{d}x,\omega)$ is well-defined for all ω in Ω_0 and by Lebesgue's theorem on dominated convergence,

$$\int_{]s,t]\times D(\epsilon,n)} x\, N(\mathrm{d}u,\mathrm{d}x,\omega) \underset{n\to\infty}{\longrightarrow} \int_{]s,t]\times D(\epsilon,\infty)} x\, N(\mathrm{d}u,\mathrm{d}x,\omega),$$

for all ω in Ω_0, i.e., almost surely. In particular

$$\int_{]s,t]\times D(\epsilon,n)} x\, N(\mathrm{d}u,\mathrm{d}x) \underset{n\to\infty}{\longrightarrow} \int_{]s,t]\times D(\epsilon,\infty)} x\, N(\mathrm{d}u,\mathrm{d}x), \quad \text{in probability},$$

and hence $(\int_{]s,t]\times D(\epsilon,n)} x\, N(\mathrm{d}u,\mathrm{d}x))_{n\in\mathbb{N}}$ is a Cauchy sequence w.r.t. convergence in probability. Now, for any n,m in $\mathbb{N}$, such that $n\le m$, we have, by Proposition 4.6 and

Corollary 4.5,

$$L\Big\{\int_{]s,t]\times D(\epsilon,m)} x\,M(\mathrm{d}u,\mathrm{d}x) - \int_{]s,t]\times D(\epsilon,n)} x\,M(\mathrm{d}u,\mathrm{d}x)\Big\}$$
$$= L\Big\{\int_{]s,t]\times D(n,m)} x\,M(\mathrm{d}u,\mathrm{d}x)\Big\}$$
$$= \Lambda\Big(L\Big\{\int_{]s,t]\times D(n,m)} x\,N(\mathrm{d}u,\mathrm{d}x)\Big\}\Big)$$
$$= \Lambda\Big(L\Big\{\int_{]s,t]\times D(\epsilon,m)} x\,N(\mathrm{d}u,\mathrm{d}x) - \int_{]s,t]\times D(\epsilon,n)} x\,N(\mathrm{d}u,\mathrm{d}x)\Big\}\Big).$$

By continuity of Λ, this shows that $(\int_{]s,t]\times D(\epsilon,n)} x\,M(\mathrm{d}u,\mathrm{d}x))_{n\in\mathbb{N}}$ is a Cauchy sequence w.r.t. convergence in probability, and hence, by completeness of $\overline{\mathcal{A}}$ w.r.t. the measure topology,

$$\int_{]s,t]\times D(\epsilon,\infty)} x\,M(\mathrm{d}u,\mathrm{d}x) := \lim_{n\to\infty}\int_{]s,t]\times D(\epsilon,n)} x\,M(\mathrm{d}u,\mathrm{d}x),$$

exists in $\overline{\mathcal{A}}$ as the limit in probability.

Finally, since $\mathcal{ID}(*)$ and $\mathcal{ID}(\boxplus)$ are closed w.r.t. weak convergence, we have that

$$\int_{]s,t]\times D(\epsilon,\infty)} x\,N(\mathrm{d}u,\mathrm{d}x) \in \mathcal{ID}(*) \quad\text{and}\quad \int_{]s,t]\times D(\epsilon,\infty)} x\,M(\mathrm{d}u,\mathrm{d}x) \in \mathcal{ID}(\boxplus).$$

Moreover, since convergence in probability implies convergence in distribution (cf. [BT, Proposition 2.20]), it follows from (6.3) and continuity of Λ that (6.2) holds.

(ii) Suppose $\int_{[-1,1]} |x|\,\nu(\mathrm{d}x) < \infty$. Then for any n in $\mathbb{N}$ and any s,t in $]0,\infty[$, such that $s<t$, we have that

$$\int_{]s,t]\times[-n,n]} |x|\,\mathrm{Leb}\otimes\nu(\mathrm{d}u,\mathrm{d}x) = (t-s)\int_{[-n,n]} |x|\,\nu(\mathrm{d}x)$$
$$= (t-s)\Big(\int_{[-1,1]} |x|\,\nu(\mathrm{d}x) + \int_{D(1,n)} |x|\,\nu(\mathrm{d}x)\Big) < \infty,$$

since ν is a Lévy measure. Hence, by Proposition 4.3, the integral $\int_{]s,t]\times[-n,n]} x\,M(\mathrm{d}u,\mathrm{d}x)$ is well-defined and, by Corollary 4.5,

$$L\Big\{\int_{]s,t]\times[-n,n]} x\,M(\mathrm{d}u,\mathrm{d}x)\Big\} = \Lambda\Big(L\Big\{\int_{]s,t]\times[-n,n]} x\,N(\mathrm{d}u,\mathrm{d}x)\Big\}\Big).$$

From this point on, the proof is exactly the same as that of (i) given above; the only difference being that the application of Lemma 6.2(i) above must be replaced by an application of Lemma 6.2(ii). ■

We are now ready to give a proof of the Lévy-Itô decomposition for free Lévy processes (in law). As is customary in the classical case (cf. [Sa]), we divide the general formulation into two parts.

6.4 Theorem. (Free Lévy-Itô I) *Let (Z_t) be a free Lévy process (in law) affiliated with a W^*-probability space $(\mathcal{A}, \tau)$, let ν be the Lévy measure appearing in the free generating triplet for $L\{Z_1\}$ and assume that $\int_{-1}^{1} |x| \ \nu(dx) < \infty$. Then (Z_t) has a representation in the form:*

$$Z_t \stackrel{\mathrm{d}}{=} \gamma t \mathbf{1}_{\mathcal{A}^0} + \sqrt{a} W_t + \int_{]0,t]\times\mathbb{R}} x \ M(\mathrm{d}u, \mathrm{d}x), \qquad (t \geq 0), \tag{6.4}$$

where $\gamma \in \mathbb{R}$, $a \geq 0$, (W_t) is a free Brownian motion in some W^*-probability space $(\mathcal{A}^0, \tau^0)$ and M is a free Poisson random measure on $(\mathsf{H}, \mathsf{B}(\mathsf{H}), \mathrm{Leb} \otimes \nu)$ with values in $(\mathcal{A}^0, \tau^0)$. Furthermore, the process

$$U_t := \int_{]0,t]\times\mathbb{R}} x \ M(\mathrm{d}u, \mathrm{d}x), \qquad (t \geq 0),$$

is a free Lévy process (in law), which is freely independent of (W_t), and the right hand side of (6.4), as a whole, is a free Lévy process (in law).

The symbol $\stackrel{\mathrm{d}}{=}$ appearing in (6.4) just means that the two operators have the same (spectral) distribution. For that reason also, it does not follow directly from (6.4) that the right hand side is a free Lévy process (in law) (contrary to the situation in the classical Lévy-Itô decomposition).

Proof of Theorem 6.4. By Proposition 2.12, we may choose a classical Lévy process (X_t), defined on some probability space $(\Omega, \mathcal{F}, P)$, such that $\Lambda(L\{X_t\}) = L\{Z_t\}$ for all t in $[0, \infty[$. Then ν is the Lévy measure for $L\{X_1\}$, so by the classical Lévy-Itô Theorem (cf. Theorem 2.16), (X_t) has a representation in the form:

$$X_t \stackrel{\text{a.s.}}{=} \gamma t + \sqrt{a} B_t + \int_{]0,t]\times\mathbb{R}} x N(\mathrm{d}u, \mathrm{d}x), \qquad (t \geq 0),$$

where (B_t) is a (classical) Brownian motion on $(\Omega, \mathcal{F}, P)$, N is a (classical) Poisson random measure on $(\mathsf{H}, \mathsf{B}(\mathsf{H}), \mathrm{Leb} \otimes \nu)$, defined on $(\Omega, \mathcal{F}, P)$ and (B_t) and N are independent. Put

$$Y_t := \int_{]0,t]\times\mathbb{R}} x \ N(\mathrm{d}u, \mathrm{d}x), \qquad (t \geq 0).$$

Now choose a free Brownian motion (W_t) in some W^*-probability space $(\mathcal{A}^1, \tau^1)$, and recall that $L\{W_t\} = \Lambda(L\{B_t\})$ for all t. Choose, further, a free Poisson random measure M on $(\mathsf{H}, \mathsf{B}(\mathsf{H}), \mathrm{Leb} \otimes \nu)$ with values in some W^*-probability space $(\mathcal{A}^2, \tau^2)$. Next, let $(\mathcal{A}^0, \tau^0)$ be the (reduced) free product of the two W^*-probability spaces $(\mathcal{A}^1, \tau^1)$ and $(\mathcal{A}^2, \tau^2)$ (cf. [VDN, Definition 1.6.1]). We may then consider $\mathcal{A}^1$ and $\mathcal{A}^2$ as two freely independent unital W^*-subalgebras of $\mathcal{A}$, such that $\tau^0_{|\mathcal{A}^1} = \tau^1$ and $\tau^0_{|\mathcal{A}^2} = \tau^2$. In particular, (W_t) and M are freely independent in $(\mathcal{A}^0, \tau^0)$.

Since $\int_{[-1,1]} |x| \ \nu(\mathrm{d}x) < \infty$, it follows from Lemma 6.3(ii) that for any t in $]0, \infty[$, the integral $U_t = \int_{]0,t]\times\mathbb{R}} x \ M(\mathrm{d}u, \mathrm{d}x)$ is well-defined, and $L\{U_t\} = \Lambda(L\{Y_t\})$. Furthermore, it follows immediately from Definition 4.1, Proposition 4.3 and Lemma 6.3 that for any t in $[0, t[$, $U_t = \int_{]0,t]\times\mathbb{R}} x \ M(\mathrm{d}u, \mathrm{d}x)$ is in the closure of $\mathcal{A}^2$ w.r.t. the measure topology. As noted in Remark 5.6, the set $\overline{\mathcal{A}^2}$ of closed, densely defined operators affiliated with $\mathcal{A}^2$ is complete (and hence closed) w.r.t. the measure topology, and therefore U_t is affiliated with $\mathcal{A}^2$ for all t. This implies, in particular, that the two processes (W_t) and (U_t) are freely independent.

Now, for any t in $]0,\infty[$, we have

$$\begin{aligned}
L\{\gamma t\mathbf{1}_{\mathcal{A}^0} + \sqrt{a}W_t + U_t\} &= \delta_{\gamma t} \boxplus D_{\sqrt{a}}L\{W_t\} \boxplus L\{U_t\} \\
&= \Lambda(\delta_{\gamma t}) \boxplus D_{\sqrt{a}}\Lambda(L\{B_t\}) \boxplus \Lambda\big(L\{Y_t\}\big) \\
&= \Lambda\big(\delta_{\gamma t} * D_{\sqrt{a}}L\{B_t\} * L\{Y_t\}\big) \\
&= \Lambda\big(L\{\gamma t + \sqrt{a}B_t + Y_t\}\big) \\
&= \Lambda(L\{X_t\}) \\
&= L\{Z_t\},
\end{aligned}$$

and this proves (6.4). We prove next that the process (U_t) is a free Lévy process (in law). For this, recall that (Y_t) is a (classical) Lévy process defined on $(\Omega,\mathcal{F},P)$ (cf. [Sa, Theorem 19.3]), and such that $L\{U_t\} = \Lambda(L\{Y_t\})$ for all t. Since (Y_t) has stationary increments, we find for any s,t in $[0,\infty[$, such that $s<t$, that

$$\begin{aligned}
L\{U_{s+t} - U_s\} &= L\Big\{\int_{]s,s+t]\times\mathbb{R}} x\, M(\mathrm{d}u,\mathrm{d}x)\Big\} = \Lambda\Big(L\Big\{\int_{]s,s+t]\times\mathbb{R}} x\, N(\mathrm{d}u,\mathrm{d}x)\Big\}\Big) \\
&= \Lambda\big(L\{Y_{s+t} - Y_s\}\big) = \Lambda(L\{Y_t\}) = L\{U_t\},
\end{aligned}$$

where we have used Lemma 6.3(ii). Thus, (U_t) has stationary increments too. Furthermore, by continuity of Λ,

$$L\{U_t\} = \Lambda\big(L\{Y_t\}\big) \xrightarrow{\text{w}} \Lambda(\delta_0) = \delta_0, \quad \text{as } t \searrow 0,$$

so that (U_t) is stochastically continuous. Finally, to prove that (U_t) has freely independent increments, consider r in $\mathbb{N}$ and $t_0, t_1, \dots, t_r$ in $[0,\infty[$, such that $0 = t_0 < t_1 < \cdots < t_r$. Then for any j in $\{1,2,\dots,r\}$ we have (cf. Lemma 6.3) that

$$U_{t_j} - U_{t_{j-1}} = \int_{]t_{j-1},t_j]\times\mathbb{R}} x\, M(\mathrm{d}u,\mathrm{d}x) = \lim_{n\to\infty} \int_{]t_{j-1},t_j]\times[-n,n]} x\, M(\mathrm{d}u,\mathrm{d}x),$$

where the limit is taken in probability. Since $\int_{]t_{j-1},t_j]\times[-n,n]} |x| \,\mathrm{Leb}\otimes\nu(\mathrm{d}u,\mathrm{d}x) < \infty$ for any n in $\mathbb{N}$ and any j in $\{1,2,\dots,r\}$, it follows from Corollary 5.5 that for any n in $\mathbb{N}$, the integrals

$$\int_{]t_{j-1},t_j]\times[-n,n]} x\, M(\mathrm{d}u,\mathrm{d}x), \qquad j = 1,2,\dots,r,$$

are freely independent operators. Hence, by Proposition 5.4, the increments

$$U_{t_1}, U_{t_2} - U_{t_1}, \dots, U_{t_r} - U_{t_{r-1}}$$

are also freely independent.

It remains to note that the right hand side of (6.4) is a free Lévy process (in law). This follows immediately from the fact that the sum of two freely independent free Lévy processes (in law) is again a free Lévy process (in law). Indeed, the stochastic continuity condition follows from the fact that addition is a continuous operation w.r.t. the measure topology, and the remaining conditions are immediate consequences of basic properties of free independence. This concludes the proof. ■

6.5 Theorem. (Free Lévy-Itô II) *Let (Z_t) be a free Lévy process (in law) affiliated with a W^*-probability space $(\mathcal{A},\tau)$ and let ν be the Lévy measure appearing in the free generating triplet for $L\{Z_1\}$. Then (Z_t) has a representation in the form:*

$$Z_t \overset{\mathrm{d}}{=} \eta t \mathbf{1}_{\mathcal{A}^0} + \sqrt{a} W_t + V_t, \qquad (t \geq 0), \tag{6.5}$$

where

- *$\eta \in \mathbb{R}$, $a \geq 0$ and (W_t) is a free Brownian in a W^*-probability space $(\mathcal{A}^0,\tau^0)$.*
- *(V_t) is a free Lévy process (in law) given by*

  $$V_t := \lim_{\epsilon \searrow 0} \Big[\int_{]0,t]\times D(\epsilon,\infty)} x \, M(\mathrm{d}u,\mathrm{d}x) - \Big(\int_{]0,t]\times D(\epsilon,1)} x \, \mathrm{Leb}\otimes\nu(\mathrm{d}u,\mathrm{d}x) \Big) \mathbf{1}_{\mathcal{A}^0} \Big], \qquad (t \geq 0),$$

  where M is a free Poisson random measure on $(\mathsf{H},\mathsf{B}(\mathsf{H}),\mathrm{Leb}\otimes\nu)$ with values in $(\mathcal{A}^0,\tau^0)$, and the limit is taken in probability.
- *(W_t) and (V_t) are freely independent processes.*

Furthermore, the right hand side of (6.5), as a whole, is a free Lévy process (in law).

Proof. The proof proceeds along the same lines as that of Theorem 6.4, and we shall not repeat all the arguments. Let (X_t) be a classical Lévy process defined on a probability space $(\Omega,\mathcal{F},P)$ such that $L\{Z_t\} = \Lambda(L\{X_t\})$ for all t. In particular, the Lévy measure for $L\{X_1\}$ is ν. Hence, by Theorem 2.16(ii), (X_t) has a representation in the form

$$X_t \overset{\text{a.s.}}{=} \eta t + \sqrt{a} B_t + Y_t, \qquad (t \geq 0),$$

where

- $\eta \in \mathbb{R}$, $a \geq 0$ and (B_t) is a (classical) Brownian motion on $(\Omega,\mathcal{F},P)$.
- (Y_t) is a classical Lévy process given by

  $$Y_t := \lim_{\epsilon \searrow 0} \Big[\int_{]0,t]\times D(\epsilon,\infty)} x \, N(\mathrm{d}u,\mathrm{d}x) - \int_{]0,t]\times D(\epsilon,1)} x \, \mathrm{Leb}\otimes\nu(\mathrm{d}u,\mathrm{d}x) \Big], \qquad (t \geq 0),$$

  where N is a (classical) Poisson random measure on $(\mathsf{H},\mathsf{B}(\mathsf{H}),\mathrm{Leb}\otimes\nu)$, defined on $(\Omega,\mathcal{F},P)$, and the limit is almost surely.

- (B_t) and (Y_t) are independent processes.

For all ϵ, t in $]0,\infty[$, we put:

$$Y_{\epsilon,t} = \int_{]0,t]\times D(\epsilon,\infty)} x \ N(\mathrm{d}u, \mathrm{d}x) - \int_{]0,t]\times D(\epsilon,1)} x \,\mathrm{Leb}\otimes\nu(\mathrm{d}u,\mathrm{d}x),$$

so that $Y_t = \lim_{\epsilon\searrow 0} Y_{t,\epsilon}$ almost surely, for each t.

As in the proof of Theorem 6.4 above, we choose, next, a W^*-probability space $(\mathcal{A}^0, \tau^0)$, which contains a free Brownian motion (W_t) and a free Poisson random measure M on $(\mathsf{H}, \mathsf{B}(\mathsf{H}), \mathrm{Leb}\otimes\nu)$, which generate freely independent W^*-subalgebras. For any ϵ in $]0,\infty[$, we put (cf. Lemma 6.3(i)),

$$V_{\epsilon,t} = \int_{]0,t]\times D(\epsilon,\infty)} x \ M(\mathrm{d}u, \mathrm{d}x) - \Big(\int_{]0,t]\times D(\epsilon,1)} x \,\mathrm{Leb}\otimes\nu(\mathrm{d}u,\mathrm{d}x)\Big)\mathbf{1}_{\mathcal{A}^0}.$$

Then for any t in $]0,\infty[$ and any ϵ_1, ϵ_2 in $]0,1[$, such that $\epsilon_1 > \epsilon_2$, we have that

$$V_{\epsilon_2,t} - V_{\epsilon_1,t} = \int_{]0,t]\times D(\epsilon_2,\epsilon_1)} x \ M(\mathrm{d}u, \mathrm{d}x) - \Big(\int_{]0,t]\times D(\epsilon_2,\epsilon_1)} x \,\mathrm{Leb}\otimes\nu(\mathrm{d}u,\mathrm{d}x)\Big)\mathbf{1}_{\mathcal{A}^0}.$$

Making the same calculation for $Y_{\epsilon_2,t} - Y_{\epsilon_1,t}$ and taking Corollary 4.5 into account, it follows that $L\{V_{\epsilon_2,t} - V_{\epsilon_1,t}\} = \Lambda(L\{Y_{\epsilon_2,t} - Y_{\epsilon_1,t}\})$. Hence, by continuity of Λ and completeness of the measure topology, we may conclude that the limit $V_t := \lim_{\epsilon\searrow 0} V_{\epsilon,t}$ exists in probability, and that $L\{V_t\} = \Lambda(L\{Y_t\})$. Moreover, as in the proof of Theorem 6.4, it follows that (W_t) and (V_t) are freely independent processes.

Now for any t in $]0,\infty[$, we have:

$$\begin{aligned} L\big\{\eta t\mathbf{1}_{\mathcal{A}^0} + \sqrt{a}W_t + V_t\big\} &= \delta_{\eta t} \boxplus D_{\sqrt{a}}L\{W_t\} \boxplus L\{V_t\} \\ &= \Lambda\big(\delta_{\eta t} * D_{\sqrt{a}}L\{B_t\} * L\{Y_t\}\big) = \Lambda\big(L\{X_t\}\big) = L\{Z_t\}. \end{aligned}$$

It remains to prove that (V_t) is a free Lévy process (in law). For this, note first that if $0 \le s < t$, we have (cf. Lemma 6.3(i)),

$$\begin{aligned} V_{s+t} - V_s &= \lim_{\epsilon\searrow 0} \big(V_{\epsilon,s+t} - V_{\epsilon,s}\big) \\ &= \lim_{\epsilon\searrow 0} \Big[\int_{]s,s+t]\times D(\epsilon,\infty)} x \ M(\mathrm{d}u, \mathrm{d}x) - \Big(\int_{]s,s+t]\times D(\epsilon,1)} x \,\mathrm{Leb}\otimes\nu(\mathrm{d}u,\mathrm{d}x)\Big)\mathbf{1}_{\mathcal{A}^0}\Big]. \end{aligned}$$

Making the same calculation for $Y_{s+t} - Y_s$, and taking Lemma 6.3(i) as well as the continuity of Λ into account, it follows that

$$L\{V_{s+t} - V_s\} = \Lambda(L\{Y_{s+t} - Y_s\}) = \Lambda(L\{Y_t\}) = L\{V_t\},$$

so that (V_t) has stationary increments. The stochastic continuity of (V_t) follows exactly as in the proof of Theorem 6.4. To see, finally, that (V_t) has freely independent increments, assume that $0 = t_0 < t_1 < t_2 < \cdots < t_r$, and consider ϵ in $]0,\infty[$. Then for any j in $\{1,2,\dots,r\}$,

$$V_{\epsilon,t_j} - V_{\epsilon,t_{j-1}} = \lim_{n\to\infty}\Big[\int_{]t_{j-1},t_j]\times D(\epsilon,n)} x\; M(\mathrm{d}u,\mathrm{d}x) - \Big(\int_{]t_{j-1},t_j]\times D(\epsilon,1)} x\,\mathrm{Leb}\otimes\nu(\mathrm{d}u,\mathrm{d}x)\Big)\mathbf{1}_{\mathcal{A}^0}\Big].$$

Hence, by Corollary 5.5 and Proposition 5.4, the increments $V_{\epsilon,t_j} - V_{\epsilon,t_{j-1}}$, $j = 1,2,\dots,r$ are freely independent, for any fixed positive ϵ. Yet another application of Proposition 5.4 then yields that the increments

$$V_{t_j} - V_{t_{j-1}} = \lim_{\epsilon\searrow 0}\big(V_{\epsilon,t_j} - V_{\epsilon,t_{j-1}}\big), \qquad (j = 1,2,\dots,r),$$

are freely independent too. ■

6.6 Remark. Let (Z_t) be a free Lévy process in law, such that $L\{Z_1\}$ has Lévy measure ν. If $\int_{[-1,1]}|x|\,\nu(\mathrm{d}x) < \infty$, then Theorems 6.4 and 6.5 provide two different "Lévy-Itô decompositions" of (Z_t). The relationship between the two representations, however, is simply that

$$\eta = \gamma + \int_{[-1,1]} x\,\nu(\mathrm{d}x) \quad\text{and}\quad V_t = U_t - t\Big(\int_{[-1,1]} x\,\nu(\mathrm{d}x)\Big)\mathbf{1}_{\mathcal{A}^0}, \qquad (t \geq 0).$$

6.7 Remark. The proof of the general free Lévy-Itô decomposition, Theorem 6.5, also provides a proof of the general existence of free Lévy processes (in law). Indeed, the conclusion of the proof of Theorem 6.5 might also be formulated in the following way: For any classical Lévy process (X_t), there exists a W^*-probability space $(\mathcal{A}^0,\tau^0)$ containing a free Brownian motion (W_t) and a free Poisson random measure M on $(\mathsf{H},\mathsf{B}(\mathsf{H}),\mathrm{Leb}\otimes\nu)$, which are freely independent, and such that

$$\begin{aligned}&\Lambda(L\{X_t\}) = \\ &\eta t\mathbf{1}_{\mathcal{A}^0} + \sqrt{a}W_t + \\ &\lim_{\epsilon\searrow 0}\Big[\int_{]0,t]\times D(\epsilon,\infty)} x\; M(\mathrm{d}u,\mathrm{d}x) - \Big(\int_{]0,t]\times D(\epsilon,1)} x\,\mathrm{Leb}\otimes\nu(\mathrm{d}u,\mathrm{d}x)\Big)\mathbf{1}_{\mathcal{A}^0}\Big], \quad (t\geq 0),\end{aligned} \tag{6.6}$$

for suitable constants η in $\mathbb{R}$ and a in $]0,\infty[$. In addition, the right hand side of (6.6) is a free Lévy process (in law) affiliated with $(\mathcal{A}^0,\tau^0)$.

Assume now that $(\nu_t)_{t\geq 0}$ is a family of distributions in $\mathcal{ID}(\boxplus)$, satisfying the two conditions

$$\nu_t = \nu_s \boxplus \nu_{t-s}, \qquad (0 \leq s < t),$$

and

$$\nu_t \xrightarrow{\mathrm{w}} \delta_0, \quad \text{as } t \searrow 0.$$

Then put $\mu_t = \Lambda^{-1}(\nu_t)$ for all t, and note that the family (μ_t) satisfies the corresponding conditions:

$$\mu_t = \mu_s * \mu_{t-s}, \qquad (0 \leq s < t),$$

and

$$\mu_t \xrightarrow{\text{w}} \delta_0, \quad \text{as } t \searrow 0,$$

by the properties of Λ^{-1}. Hence, by the well-known existence result for classical Lévy processes, there exists a classical Lévy process (X_t), such that $L\{X_t\} = \mu_t$ and hence $\Lambda(L\{X_t\}) = \nu_t$ for all t. Therefore, the right hand side of (6.6) is a free Lévy process (in law), (Z_t), such that $L\{Z_t\} = \nu_t$ for all t.

The above argument for the existence of free Lévy processes (in law) is, of course, based on the existence of free Poisson random measures proved in Theorem 3.3. The existence of free Lévy processes (in law) can also, as noted in [Bi] and [Vo3], be proved directly by a construction similar to that given in the proof of Theorem 3.3. The latter approach, however, is somewhat more complicated than the construction given in the proof of Theorem 3.3, since, in the general case, one has to deal with unbounded operators throughout the construction, whereas free Poisson random measures only involve bounded operators.

References

[BGS] A. Ben Ghorbal and M. Schürmann, *Non-commutative notions of stochastic independence*, Math. Proc. Cambridge Philos. Soc. **133** (2002), 531-561.

[Bi] P. Biane, *Processes with free increments*, Math. Zeitschrift **227** (1998), 143-174.

[BP] H. Bercovici and V. Pata, *Stable laws and domains of attraction in free probability theory*, Ann. Math. **149** (1999), 1023-1060.

[BT] O.E. Barndorff-Nielsen and S. Thorbjørnsen, *Selfdecomposability and Lévy processes in free probability*, Bernoulli **8** (2002), 323-366.

[BV] H. Bercovici and D.V. Voiculescu, *Free Convolution of Measures with Unbounded Support*, Indiana Univ. Math. J. **42** (1993), 733-773.

[KR] R.V. Kadison and J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras, vol I-II*, Graduate Studies in Math. 15, American Mathematical Society (1997).

[Mu] N. Muraki, *The five independences as quasi universal products*, Inf. Dim. Anal., Quant. Probab. and Rel. Topics **5** (2002), 113-134.

[Ne] E. Nelson, *notes on non-commutative integration*, J. Funct. Anal. **15** (1974), 103-116.

[Pe] Gert K. Pedersen, *Analysis Now*, Graduate Texts in Mathematics **118**, Springer Verlag (1989).

[Pa] V. Pata, *Domains of partial attraction in non-commutative probability*, Pacific J. Math. **176** (1996), 235-248.

[Sa] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Studies in Advanced Math. **68** (1999).

[Sc] M. Schürmann, *Direct sums of tensor products and non-commutative independence*, J, Funct. Anal. **133** (1995), 1-9.

[Se] I.E. SEGAL, *A non-commutative extension of abstract integration*, Ann. Math. **57** (1953), 401-457; correction **58** (1953), 595-596.

[Sp] R. SPEICHER, *Universal products*, in D.V. Voiculescu (ed.), *Free Probability Theory*, Fields Institute Communications **12**, 257-266.

[St] W.F. STINESPRING, *Integration theory for gages and duality for unimodular groups*, Transactions of the AMS. **90** (1959), 15-56.

[Vo1] D.V. VOICULESCU, *Addition of certain non-commuting random variables*, J. Funct. Anal. **66**, (1986), 323-346.

[Vo2] D.V. VOICULESCU, *Circular and semicircular systems and free product factors*, "Operator Algebras, Unitary Representations, Algebras, and Invariant Theory", *Progress in Math.* Vol. 92, Birkhäuser, 1990, 45–60.

[Vo3] D.V. VOICULESCU, *Lectures on Free Probability*, Lecture notes from the 1998 Saint-Flour Summer School on Probability Theory.

[VDN] D.V. VOICULESCU, K.J. DYKEMA AND A. NICA, *Free Random Variables*, American Mathematical Society, Providence, RI (1992).

Department of Mathematical Sciences
University of Aarhus
Ny Munkegade
DK-8000 Aarhus C
Denmark
oebn@imf.au.dk

Department of Mathematics and Computer Science
University of Southern Denmark
Campusvej 55, 5230 Odense M
Denmark
steenth@imada.sdu.dk