A new direct method for reconstructing isotropic conductivities in the plane

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Abstract. In this paper we describe a new, direct and mathematically exact method for the reconstruction of the isotropic conductivity in a plane body from static electric measurements on the boundary of the body. The method is inspired by a uniqueness proof for the inverse conductivity problem due to Brown–Uhlmann and covers conductivities having essentially one derivative. Moreover, we give a numerical implementation of the algorithm and test the performance on a simple, synthetic example.

Keywords: electrical impedance tomography, inverse conductivity problems, reconstruction method, $\overline{\partial}$-method.

1. Introduction

The inverse conductivity problem is the mathematical formulation of the general problem in Electrical Impedance Tomography (EIT), i.e. the imaging of an unknown conductivity distribution in a body from static electric measurements on the boundary of the body. To pose the mathematical problem we introduce the bounded and smooth set $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$, which models the body we want to image. The boundary of the body is denoted by $\partial\Omega$. We assume that the conductivity distribution inside Ω is isotropic and hence described by a strictly positive and bounded function γ defined in Ω. For simplicity we assume that $\gamma = 1$ near $\partial\Omega$. In the mathematical language it is convenient to describe the boundary measurements by functions, which are continuously defined on $\partial\Omega$ (we do not attempt to model electrode configurations etc.). The experiments are performed by inducing a voltage potential f on $\partial\Omega$ and then measuring the resulting current flux through the boundary. This quantity is given by

$$g = (\nu_1, \nu_2) \cdot (\nabla u)|_{\partial\Omega},$$

where (ν_1, ν_2) is the outward unit normal to $\partial\Omega$ and u is the voltage potential inside Ω defined as the unique solution to

$$\nabla \cdot \gamma \nabla u = 0, \text{ in } \Omega, \quad u = f, \text{ on } \partial\Omega. \tag{1}$$

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Taking all possible boundary fields f and then measuring the resulting current fluxes g gives the knowledge of the so-called Dirichlet-to-Neumann map or voltage-to-current map Λ_γ defined by

$$\Lambda_\gamma f = g.$$

This map encodes all possible boundary measurements, which can in principle be considered for EIT.

The inverse conductivity problem, as it was posed in (Calderón 1980), consists of two parts. First, the question of uniqueness: if for two conductivity distributions (γ_1, γ_2) all boundary measurements agree $(\Lambda_{\gamma_1} = \Lambda_{\gamma_2})$, can we then conclude that the conductivities must be the same $(\gamma_1 = \gamma_2)$? Second, the question of reconstruction: assuming that uniqueness holds, how should we compute the conductivity γ from the boundary measurements Λ_γ?

For the three-dimensional problem the uniqueness question was answered affirmatively in (Sylvester & Uhlmann 1987) for conductivities having essentially two derivatives, and a reconstruction algorithm was given independently in (Nachman 1988) and (Novikov 1988). For the two-dimensional problem the uniqueness and reconstruction questions were solved in (Nachman 1996) for conductivities having essentially two derivatives. The method of proof was based on the $\overline{\partial}$-method of inverse scattering. The reconstruction algorithm has since been implemented numerically (Siltanen, Mueller & Isaacson 2000, Siltanen, Mueller & Isaacson 2001a, Siltanen, Mueller & Isaacson 2001b, Mueller & Siltanen n.d.). More recently in (Brown & Uhlmann 1997) an affirmative answer was given to the uniqueness question for the two-dimensional problem for conductivities having essentially one derivative. Again the $\overline{\partial}$-method of inverse scattering was the essential tool. Several extensions of these results exist in the literature (see (Uhlmann 1999)), but surprisingly enough only little is known, when the conductivity is not assumed to have a certain amount of smoothness. The general problem of unique determination and reconstruction of an arbitrary (discontinuous) conductivity still remains to be solved.

In this paper we will consider the reconstruction issue for the two-dimensional problem and discuss a recent direct reconstruction method obtained jointly in (Knudsen & Tamasan 2001). The algorithm is based on the uniqueness proof of Brown and Uhlmann and gives a mathematically exact way to reconstruct conductivities, which are assumed to have essentially one derivative. In some sense the algorithm generalizes Nachman's algorithm; in particular the method attacks directly the full nonlinear problem, and hence it is not based on linearization or iterative schemes. For these other types of algorithms we refer to (Cheney, Isaacson & Newell 1999) and (Borcea 2002).

In the next section we will explain the theory behind the reconstruction algorithm. Then in section 3 we will give a numerical implementation of the algorithm, and finally in section 4 we will test the algorithm on a synthetic example.

2. The reconstruction algorithm

Extend the conductivity outside Ω by $\gamma = 1$ and let u be a solution to the conductivity equation (1) in $\mathbb{R}^2$. Introduce the vector $(v, w) = \gamma^{1/2}(\partial u, \overline{\partial} u)$, where

$$\partial = \frac{\partial}{\partial z} = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2}), \quad \overline{\partial} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2}). \tag{2}$$

Here and in the sequel we identify the complex number $z = x_1 + ix_2 \in \mathbb{C}$ and $(x_1, x_2) \in \mathbb{R}^2$. Since u solves (1) it follows that (v, w) solves the system

$$\overline{\partial} v = qw, \quad \partial w = \overline{q} v, \tag{3}$$

where $q = -\gamma^{-1/2}\partial\gamma^{1/2}$. Note that q is well-defined when γ admits one derivative. Note further that q is supported inside Ω, i.e. $q = 0$ in $\mathbb{R}^2 \setminus \Omega$.

The idea is now to apply the $\overline{\partial}$-method of inverse scattering to the system (3). This method was introduced by Beals and Coifman in a series of papers; the analysis of (3) was given in (Beals & Coifman 1985, Beals & Coifman 1988). For any fixed $k \in \mathbb{C}$ we look for a special exponentially growing solution $\Psi(z, k) = (\Psi_1, \Psi_2)(z, k)$ to the system (3), which has the asymptotics

$$e^{-izk}\Psi(z, k) \to (1, 0) \text{ as } |z| \to \infty. \tag{4}$$

To elaborate on this definition let $e(z, k) = \exp(i(zk + \overline{zk})) = \exp(i2\mathrm{Re}(zk))$ and introduce $m(z, k) = e^{-izk}\Psi(z, k)$ and the linear combination

$$m_\pm(z, k) = m_1(z, k) \pm \overline{m_2(z, k)} e(z, -k). \tag{5}$$

Since Ψ satisfies (3) and has the asymptotics (4), $m_\pm$ satisfies

$$\overline{\partial} m_\pm(z, k) = \pm q(z) e(z, -k) \overline{m_\pm(z, k)}$$

together with the asymptotic condition $m_\pm(z, k) \to 1$ as $|z| \to \infty$. Equivalently $m_\pm$ satisfies the weakly singular integral equation

$$m_\pm(z, k) = 1 \pm \frac{1}{\pi} \int_\Omega \frac{q(z') e(z', -k)}{z - z'} \overline{m_\pm(z', k)} dx'_1 dx'_2, \tag{6}$$

which can be solved uniquely for $m_\pm$ in a certain function space.

Associated with the system (3) is then the function

$$S(k) = -\frac{i}{\pi} \int_\Omega e(z, k) \overline{q}(z) m_1(z, k) dx_1 dx_2, \tag{7}$$

the so-called non-physical scattering transform of the potential q. Even though the scattering transform is not observable in experiments, it is very useful as an intermediate object when solving the inverse problem. The usefulness is two-fold. First, the scattering transform can be computed from the boundary measurements, and second, the conductivity can be computed from the scattering transform. This gives a reconstruction procedure consisting of the two steps

$$\Lambda_\gamma \to S \to \gamma.$$

This decomposition was used first in (Brown & Uhlmann 1997), who proved that Λ_γ determines S uniquely and that S determines γ uniquely. Also the stability analysis in

(Barceló, Barceló & Ruiz 2001) is based on the decomposition. They showed essentially that the first step is unstable, while the second step is stable.

We will in the following two subsections more carefully describe each step in this reconstruction procedure. For complete proofs we refer to (Knudsen & Tamasan 2001).

2.1. From Λ_γ to S

Since Ψ solves (3) and q is supported in Ω, integrating by parts in (7) shows that

$$S(k) = -\frac{i}{2\pi}\int_{\partial\Omega} e^{i\overline{zk}}\overline{\nu}(z)\Psi_2(z,k)dS, \quad k \in \mathbb{C}, \tag{8}$$

where $\overline{\nu}(z) = \nu_1(z) - i\nu_2(z) \in \mathbb{C}$ and dS is the surface element of the boundary. Hence to compute S from Λ_γ we have to compute the boundary values of Ψ. To do so we will derive equations on $\partial\Omega$, which describe $\Psi|_{\partial\Omega}$ uniquely.

First, we consider the behavior of Ψ outside Ω. Define the exponentially growing Green's function g_k for $\overline{\partial}$ by

$$g_k(z) = \frac{1}{\pi}\frac{e^{-ikz}}{z}.$$

Let $z \in \partial\Omega$ and define the single layer potentials $\mathcal{S}_k$, $\overline{\mathcal{S}}_k$ by

$$\mathcal{S}_k f(z) = p.v.\int_{\partial\Omega} f(\zeta)g_k(\zeta - z)d\zeta, \quad \overline{\mathcal{S}}_k f(z) = p.v.\int_{\partial\Omega} f(\zeta)\overline{g_k(\zeta - z)}d\zeta.$$

Here *p.v.* (principal value) indicates that the integral is given meaning by a certain limiting procedure. the important observation is now that any vector Φ, which is a solution to (3) outside Ω and has exponential growth described by $e^{-izk}\Phi \to (1,0)$ for $|z| \to \infty$, must satisfy the equations

$$(I - i\mathcal{S}_k)\Phi_1 = 2e^{izk}, \quad (I + i\overline{\mathcal{S}_k})\Phi_2 = 0 \tag{9}$$

on $\partial\Omega$. In particular $\Psi|_{\partial\Omega}$ satisfies (9).

Second, we consider the behavior inside Ω. Let ∂_τ be the derivative along $\partial\Omega$ and let ∂_τ^{-1} be its inverse (defined on a suitable space up to an additive constant). Define $H_\gamma = \Lambda_\gamma\partial_\tau^{-1}$. Then any vector Φ, which solves (3) inside Ω, must satisfy the equation

$$(iH_\gamma - I)\nu\Phi_1 = (iH_\gamma + I)\overline{\nu}\Phi_2 \tag{10}$$

on $\partial\Omega$. Again $\Psi|_{\partial\Omega}$ is one such solution.

It can now be shown that $\Psi|_{\partial\Omega}$ is the only function that satisfies both (9) and (10) (see (Knudsen & Tamasan 2001)). Hence to find $\Psi_2|_{\partial\Omega}$ we have to solve (9)–(10) simultaneously, and then insert the function into (8). This gives a method for computing S from Λ_γ.

2.2. From S to γ

To compute γ from S the crucial observation is that with respect to the variable k, $m(z,k)$ satisfies a differential equation. Indeed, the function $m^\pm(z,k) = m_1(z,k) \pm m_2(z,k)$ can for any fixed $z \in \Omega$ be seen to satisfy

$$\overline{\partial}_k m^\pm(z,k) = \pm S(k)e(z,-k)\overline{m^\pm(z,k)}$$

and the asymptotic condition $m^{\pm}(z,k) \to 1$ for $|k| \to \infty$. Here $\overline{\partial}_k$ denotes the derivative with respect to the complex variable $\overline{k}$, cf. (2). Equivalently $m^{\pm}$ satisfies the integral equation

$$m^{\pm}(z,k) = 1 \pm \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{S(k')e(z,-k')}{k-k'} \overline{m^{\pm}(z,k')} dk'_1 dk'_2,$$

which is uniquely solvable (note the similarity between this equation and (6)). This gives a method for the computation of m from S. Eventually, we can then from m compute q and γ. A theoretical result shows, however, that if we instead compute the solution $\tilde{m}^+$ to

$$\tilde{m}^+(z,k) = 1 + \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\overline{S(-k')}e(z,-k')}{k-k'} \overline{\tilde{m}^+(z,k')} dk'_1 dk'_2, \tag{11}$$

then we simply find

$$\gamma(z) = (\mathrm{Re}(\tilde{m}^+(z,0)))^2. \tag{12}$$

Hence by knowing S, we can solve (11) and then obtain the conductivity γ from (12).

3. Implementation of the algorithm

In this section we give a numerical implementation of the algorithm described in section 2. The special example we will analyze below in section 4 is a radial conductivity on the ball $B(0,1) = \{z \in \mathbb{R}^2 \mid |z| \leq 1\}$, i.e. a conductivity γ with $\gamma(z) = \gamma(|z|)$. We will occasionally take this into account in the design of the implementation

3.1. Computing the scattering transform from boundary data

As noted in (Sylvester 1992) in the case of a radial conductivity on $B(0,1)$, the eigenfunctions of the Dirichlet-to-Neumann map is the Fourier basis $e^{in\theta}$, $n \in \mathbb{Z}$, i.e.

$$\Lambda_\gamma e^{in\theta} = \lambda_n e^{in\theta}.$$

where the eigenvalues $\{\lambda_n\}$ are positive real numbers. Thus the Dirichlet-to-Neumann map is conveniently represented by its eigenvalues. Also for the operator ∂_τ^{-1} is easily represented in a Fourier basis represented

$$\partial_\tau^{-1} e^{in\theta} = \frac{1}{in} e^{in\theta}, \quad n \neq 0.$$

Hence the operator H_γ can be implemented in a Fourier basis by

$$H_\gamma e^{in\theta} = \frac{\lambda_n}{in} e^{in\theta}.$$

This formula can be used in setting up a matrix for the linear operator H_γ, which acts on discrete functions on the boundary.

The equations (9)–(10) are a complete characterization of $\Psi|_{\partial\Omega}$. To compute $\Psi|_{\partial\Omega}$ from this equation numerically, a suggestion could be to discretize the boundary operators involved as a matrix A and the right hand side in the equation as a vector

b, and then solve the linear system $Ax = b$ numerically. This idea has in preliminary tests caused difficulties, due to the fact that when the equations (9)–(10) are solved numerically, the exponentially growing part $\Psi_1|_{\partial\Omega}$ will dominate the solution. However, the computation of S by (8) involves only $\Psi_2|_{\partial\Omega}$, which is small and dominated by computational errors.

To overcome the difficulty we suggest to use the asymptotic value $\Psi_1(z,k) \approx e^{izk}$ in the equation (10) and hence consider the equation

$$(iH_\gamma + I)\overline{\nu}\Psi_2^a = (iH_\gamma - I)\nu e^{izk} \tag{13}$$

for an approximation Ψ_2^a to Ψ_2. This linear equation can easily be solved numerically (in the least squares sense).

Having found the approximation $\Psi_2^a|_{\partial\Omega}$, we can by integrating along the boundary compute an approximation of the scattering transform by

$$S^a(k) = -\frac{i}{2\pi}\int_{\partial\Omega} e^{i\overline{zk}}\Psi_2^a(z,k)\overline{\nu}(z)dS. \tag{14}$$

3.2. Computing the conductivity from the scattering transform

In section 2 a weakly singular integral equation appears both in the forward problem (6) and in the inverse problem (11). In our numerical test solving (6) is necessary in the computation of the exact S from the known potential, while solving (11) is an essential step in the reconstruction algorithm. Both equations have the generic form

$$m(v,p) = 1 + \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{T(v',p)}{v-v'}\overline{m}(v',p)d\mu(v'), \tag{15}$$

where v is the variable (z, k, respectively), p is the parameter (k, z, respectively), $T : \mathbb{C}\times\mathbb{C} \mapsto \mathbb{C}$ is the multiplier in the equation ($\pm q(z)e(z,-k)$, $\overline{S(-k)}e(z,-k)$, respectively). In the following we will suppress the dependency on the parameter p.

To solve (15) numerically we adapt a general method described in (Vainikko 1993) for solving such integral equations. This method has previously been used for solving Lippmann-Schwinger type equations numerically, see (Vainikko 2000, Mueller & Siltanen n.d., Hohage 2001). We will only sketch the method; for complete details and convergence analysis we refer to the forthcoming paper (Knudsen, Mueller & Siltanen 2003).

We will assume that the coefficient T is compactly supported inside an open set D. In the forward problem (6) this assumption holds, since q is compactly supported, but in the inverse problem the scattering transform S is not of compact support. In that case we will have to cut-off the scattering transform by putting $S(k) = 0$ for $|k| > R$ sufficiently large. This cut-off introduces a systematic error, which, however, can be neglected if R is large.

The solution m is computed on the support D of T. Assume for simplicity that $D = [-1,1]^2$ (the general case can be treated similarly). Let n be a positive integer and let

$$\mathbb{Z}_n^2 = \{(z_1,z_2)\in\mathbb{Z}^2 \mid -2^{n-1} \le z_j \le 2^{n-1}, j = 1,2\}.$$

Fix the discretization level $h = 2^{1-n}$. Then $h\mathbb{Z}_n^2$ is a uniform grid on D consisting of $(2^n+1)^2$ points. The grid approximation of a function f on D is defined by

$$f_{j,h} = f(jh), \quad j \in \mathbb{Z}_n^2.$$

Then (15) is approximated by the matrix equation

$$m_{j,h} + (\mathbf{T}_h \overline{m}_{\cdot,h})_{j,h} = 1, \quad j \in \mathbb{Z}_n^2,$$

where

$$(\mathbf{T}_h \phi_{\cdot,h})_{j,h} = \sum_{k \in \mathbb{Z}_n^2,\, j \neq k} g_{j-k,h}(T_{k,h}\phi_{k,h}),$$

and $g(z) = 1/(\pi z)$. The advantage of this implementation is that the convolution structure of the operator is preserved. Hence $\mathbf{T}_h$ can be implemented using fast Fourier transform, i.e. by computing as matrices

$$(\mathbf{T}_h \phi_{\cdot,h})_{\cdot,h} = \mathrm{iFFT}(\mathrm{FFT}(g_{\cdot,h}) .* \mathrm{FFT}(T_{\cdot,h}\phi_{\cdot,h}))$$

where $.*$ denotes pointwise multiplication of matrices. This method gives a complexity $\mathcal{O}(N^2 \log N)$ for one application of the convolution operator. Having implemented the convolution operator, the equation can be solved numerically using an iterative linear solver. It can can be shown (Knudsen et al. 2003) that the convergence rate for the particular implementation described here is $\mathcal{O}(h)$. Concerning computational speed and accuracy we note that in (Vainikko 2000), a multi-grid extension of the general solution method is described. The implementation and analysis of this method for the particular equation (15) is given in (Knudsen et al. 2003).

4. Numerical results

Our test example is the radial conductivity

$$\gamma(z) = \begin{cases} (1 + 10F(|z|)), & |z| < 3/4, \\ 1, & |z| \geq 3/4, \end{cases}$$

where

$$F(t) = |t^2 - \rho^2|^{1.1} |t^2 - \rho^2/4|^{1.1} \cos(4\pi t/3).$$

This defines a sufficiently regular conductivity with $\gamma(z) = 1$ for $|z| > 3/4$. A plot of the test conductivity and the associated potential can be seen in Figure 1.

For this conductivity we compute the eigenvalues of the Dirichlet-to-Neumann map. In (Siltanen et al. 2000) a direct method for doing so is given. The idea is to approximate a radial conductivity from below and above by piecewise constant conductivities and then compute by explicit formulas the eigenvalues for the Dirichlet-to-Neumann associated the approximate conductivities. This gives upper and lower bounds for the eigenvalues λ_n, which are then approximated by taking the average of the bounds. We have used this method in the computation of the eigenvalues for our specific example.

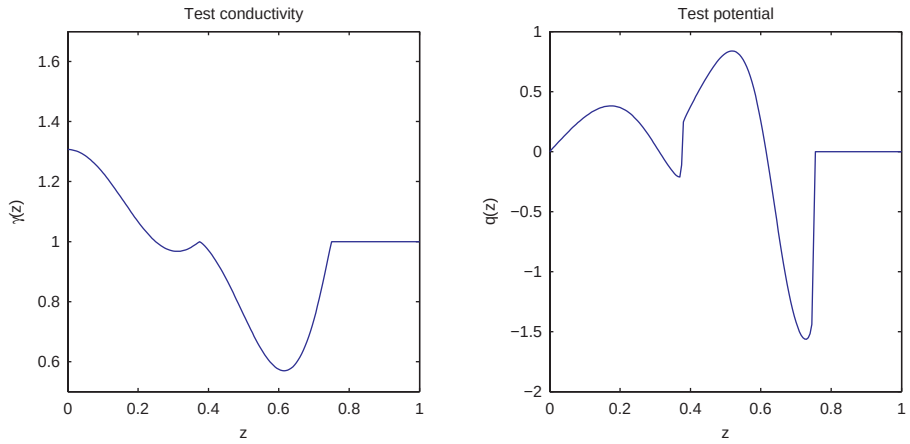

Figure 1. Profile of the conductivity and potential.

We first compute the scattering transform by solving the forward problem. To compute S from q we solve the equation (6) for $m_\pm$ and the linear system (5) for m_1, and then we integrate in (7). In the implementation we use the method for solving integral equations equations of the form (15) numerically outlined above. We have chosen the discretization parameter $n = 7$. In Figure 2 the scattering transform along the real axis is displayed. Given the exact scattering transform we can solve (11) and

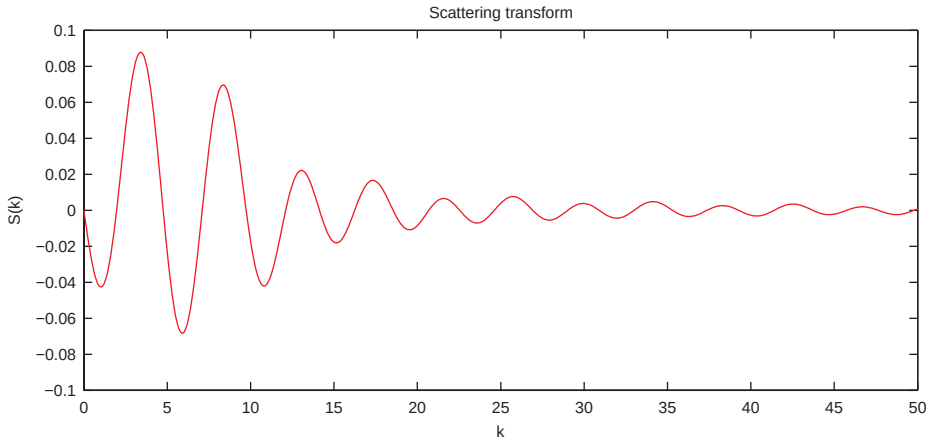

Figure 2. The true scattering transform along the axis $k > 0$.

then compute the conductivity using (12). The result of this inversion is displayed in Figure 3 together with the true conductivity. Moreover, we have plotted the relative error $|\gamma(x) - \gamma_{\rm rec}(x)|/|\gamma(x)|$. We see that the reconstruction is very accurate. This shows that the difficulty part in the reconstruction algorithm is to compute from the boundary measurements a good approximation of the scattering transform.

Next, we consider the approximation $S^{\rm a}$. From the precomputed eigenvalues we have computed a matrix version of H_γ. Then we solved (13) for Ψ_2^a and finally computed S^a

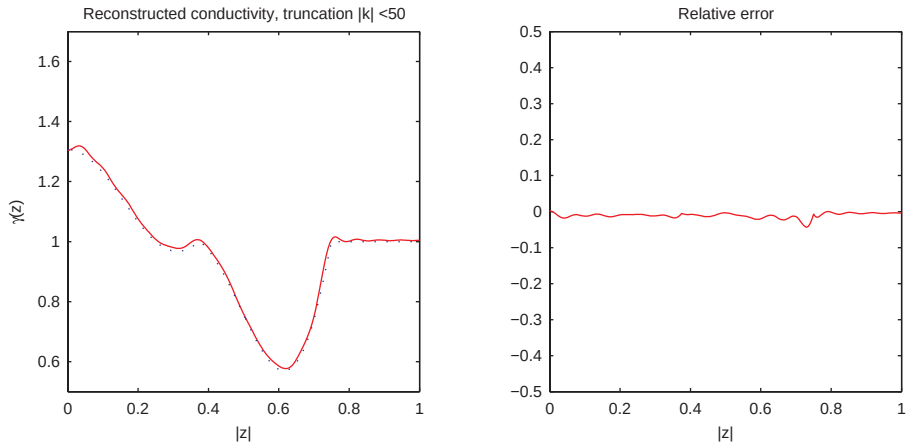

Figure 3. The reconstructed conductivities. Dotted curve is the true conductivity and solid curve is the conductivity reconstructed from $S(k),\ k < 50$.

by integrating numerically in (14). In Figure 4 we have displayed $S^{\mathrm{a}}(k)$ together with the true scattering transform S. Moreover, the error $S(k) - S^{\mathrm{a}}$ is displayed. The axis is truncated at $k = 20$, since S^{a} becomes highly inaccurate beyond $k = 15$ due to numerical errors. We note that below this value the approximation seems accurate. To

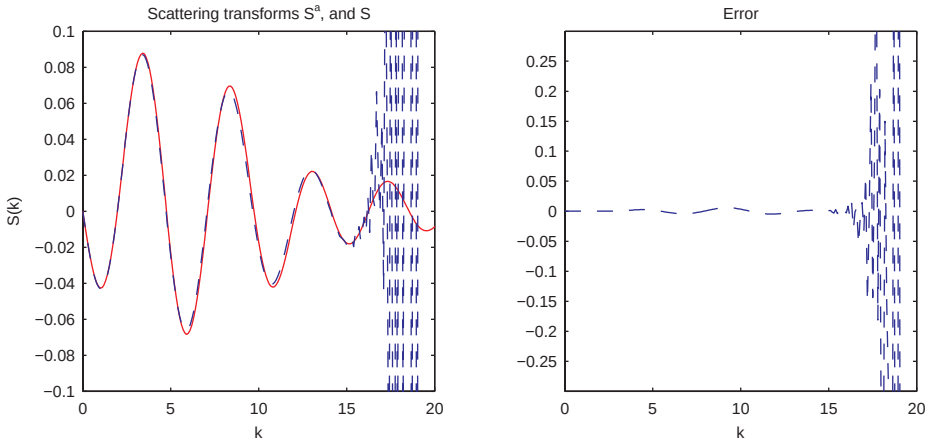

Figure 4. The scattering transforms S (solid) and S^{a} (dashed).

reconstruct from the approximate scattering transform S^{a}, we will have to truncate this function before doing the inversion. This truncation can be seen as a regularization strategy (Mueller & Siltanen n.d.), and to illustrate this effect for our example, we have chosen to truncate at the values $k = 10, 15$, and then reconstruct from the truncated S^{a}. In Figure 5 – 6 the reconstructions are displayed together with the true conductivity. Furthermore, we have for comparison displayed the reconstructed conductivity based on the same truncation of the true S. We see from the figures that when the $k = 10$, then the inversion gives a low frequency approximation of the conductivity. The quality of

the reconstruction obtained from S^a is comparable to the one obtained from S. When we cut-off at $k = 15$, the reconstruction is more accurate.

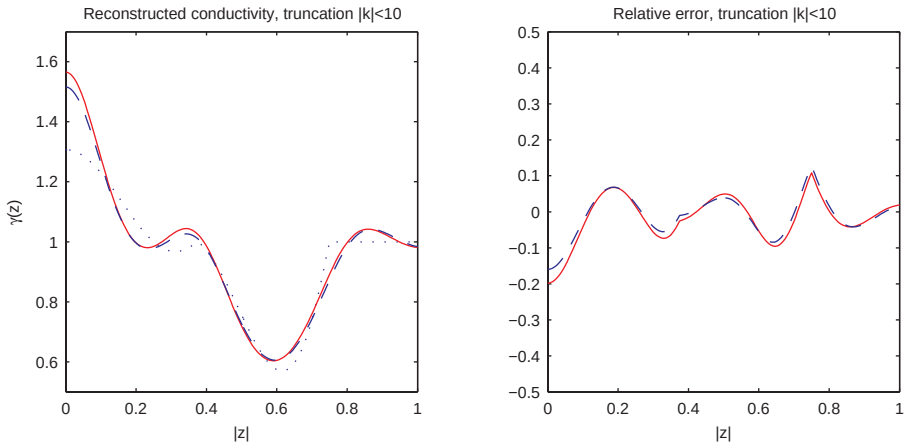

Figure 5. The reconstructed conductivities. Dashed curve is the conductivity reconstructed from $S^{\mathrm{a}}(k),\ k < 10$, solid curve is the conductivity reconstructed from the true $S,\ k < 10$, and dotted curve is the true conductivity.

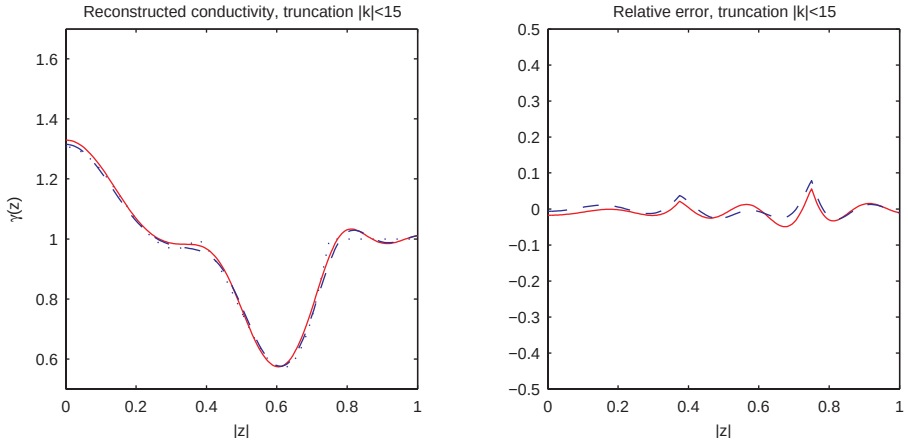

Figure 6. The reconstructed conductivities. Dashed curve is the conductivity reconstructed from $S^{\mathrm{a}}(k),\ k < 15$, solid curve is the conductivity reconstructed from the true $S,\ k < 15$, and dotted curve is the true conductivity.

5. Conclusion

In this paper we have described a direct and mathematically exact method for the reconstruction of conductivities from boundary measurements. Moreover, we have given a numerical implementation of the algorithm, which have been tested on a simple example. The conclusions are as follows:

Concerning the computation of the scattering transform from the Dirichlet-to-Neumann map, it is not yet clear how to compute most accurately the function Ψ_2 from the equations (9)–(10). The suggested method based on solving (13) seems to give a reasonable approximation S^a to the scattering transform, but a better understanding of the approximation is needed. An important issue in this context concerns the choice of cut-off parameter for truncating the scattering transform before the inversion is done. We have seen that this choice has a great impact on the reconstructions.

The numerical solution of the integral equation (15) is important both when computing the scattering transform from a known potential and when solving the inverse problem. The implementation given here is relatively fast and accurate (though not real-time), and it shows that this part of the algorithm is stable.

It is too soon to draw any definite conclusions concerning the practical value of the algorithm. The results based on a very simple test example are promising, and it will be interesting to see how well the algorithm performs on more realistic examples and eventually on real data.

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