Normal modified stable processes

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Abstract

This paper discusses two classes of distributions, and stochastic processes derived from them: modified stable (MS) laws and normal modified stable (NMS) laws. This extends corresponding results for the generalised inverse Gaussian (GIG) and generalised hyperbolic (GH) or normal generalised inverse Gaussian ($NGIG$) laws. The wider framework thus established provides, in particular, for added flexibility in the modelling of the dynamics of financial time series, of importance especially as regards OU based stochastic volatility models for equities. In the special case of the tempered stable OU process an exact option pricing formula can be found, extending previous results based on the inverse Gaussian and gamma distributions.

Contents

Keywords: Lévy process; Inverse Gaussian; OU process; Stable; Stochastic volatility; Subordination; Tempered stable.

1. Introduction

This paper discusses extensions of the concept of Normal Inverse Gaussian processes, or NIG processes for short, to what we shall call *Normal Modified Stable processes*, or NMS processes.

The term NIG processes, as used here, indicates a class of stochastic processes introduced and studied in Barndorff-Nielsen (1997), Barndorff-Nielsen (1998a), Barndorff-Nielsen (1998b), Barndorff-Nielsen and Shephard (2001a), Barndorff-Nielsen and Shephard (2001b), Barndorff-Nielsen and Shephard (2001c) and Barndorff-Nielsen and Levendorskii (2001), see also Eberlein (2000), Eberlein and Prause (2000), Prause (1998), Tompkins and Hubalek (2000), Barndorff-Nielsen and Prause (2001). As discussed in the papers cited and in references given there, the family of NIG (normal inverse Gaussian) distributions and the NIG processes, which are constructed around the NIG family, have been found to provide accurate modelling of a great variety of empirical findings in the physical sciences and in financial econometrics. The wider class of $NGIG$ (Normal Generalised Inverse Gaussian) or GH (Generalised Hyperbolic) processes (cf. the references cited above) provides additional possibilities for realistic modelling of dynamical phenomena.

Still, it is of some interest, and mathematically natural, to generalise a step further. In particular this will establish a more flexible modelling framework and allow for additional testing of NIG based models. The generalisation we shall be discussing is based on an extension of the family of generalised inverse Gaussian (GIG) distributions to a class of distributions on $\mathbf{R}_+ = (0, \infty)$, the *Modified Stable* or MS laws. The MS laws come about in the same way that the GIG laws are derived from the IG (inverse Gaussian), namely by exponential and power tempering (or tilting) from one of the positive κ-stable ($0 < \kappa < 1$) laws. Using the MS distributions as mixing distributions for normal variance-mean mixtures yields the class of NMS (normal modified stable) laws.

We surmise that all the MS distributions are infinitely divisible, and in fact selfdecomposable, but have not, so far, been able to show this in general. As is well known, the GIG distributions are selfdecomposable and the same is true of the subclass TS of MS obtained by exponential tempering alone.

The NTS laws, i.e. normal variance-mean mixtures with TS mixing, are of some special interest because the NTS Lévy processes generated from them, which may also be viewed as subordinations of Brownian motion with drift by the TS subordinators, are "structure preserving with respect to convolution" in the same sense as is the case for the NG (normal gamma) and the NIG Lévy processes.[1]

In Section 2 we introduce the class of MS laws and discuss some of their properties. Section 3 provides some background material on subordination, and in Sections 4 and 5 we consider the special case of the TS and NTS laws and Lévy processes. In particular, we determine the tail behaviour of the NTS laws and we derive an expression for the Lévy density from which the small jumps regimes of the NTS Lévy processes can be inferred. Section 6 indicates the relation to NG (*Normal Gamma*) and $RLPE$ Lévy processes (*Regular Lévy Processes of Exponential type*).

The relation of the results derived to the OU based modelling approach, discussed in Barndorff-Nielsen and Shephard (2001a), Barndorff-Nielsen and Shephard (2001b) and Barndorff-Nielsen and Shephard (2001c), is considered in Sections 7-9. Section 7 sets the scene, as it were, and in Section 8 procedures for efficient simulation from the models, using recent work of Rosinski (2001), are outlined. An application to a financial data set is given in Section 9, and the final Section 10 contains a few concluding remarks. In two Appendices A and B we recall, for easy reference, the definitions and some properties of the generalised inverse Gaussian and generalised hyperbolic distributions.

2. MS and NMS laws

Let $p(x;\kappa,\delta)$ denote the probability density function of the positive κ-stable law $S(\kappa,\delta)$ with cumulant transform $-\delta(2\theta)^\kappa$, $0<\kappa<1$, and let $p(x;\kappa,\delta,\gamma)$ denote the exponentially tilted version of $p(x;\kappa,\delta)$ defined by

$$p(x;\kappa,\delta,\gamma) = e^{\delta\gamma} p(x;\kappa,\delta) e^{-\frac{1}{2}\gamma^{1/\kappa}x}. \tag{2.1}$$

The distribution with density (2.1) ($\kappa \in (0,1), \delta > 0, \gamma \geq 0$) will be referred to as a *tempered stable law* and we denote it by $TS(\kappa,\delta,\gamma)$. Next, consider for any $\nu \in \mathbf{R}$ and $\gamma \vee (-\nu) > 0$ the derived probability density

$$p(x;\kappa,\nu,\delta,\gamma) = c(\kappa,\nu,\delta,\gamma) x^{\nu+\kappa} p(x;\kappa,\delta,\gamma), \tag{2.2}$$

where $c(\kappa,\nu,\delta,\gamma)$ is a norming constant.

[1]The class of TS distributions was introduced by Tweedie (1984). Hougaard (1986) discussed their applicability in survival analysis. See also Jørgensen (1987) and Brix (1999).

In Geman, Madan, and Yor (2000) what are here called NTS Lévy processes have been studied, in the case of zero drift, from a viewpoint different from the one of the present paper.

Asymptotically as $x \to \infty$ the density $p(x;\kappa,\delta,\gamma)$ is of the order of $x^{-1-\nu}\exp(-\gamma^2 x/2)$. Hence $x^{\nu+\kappa}p(x;\kappa,\delta,\gamma)$ is certainly integrable at $+\infty$ under the stated condition on γ and ν.

To see that $x^{\nu+\kappa}p(x;\kappa,\delta,\gamma)$ is, in fact, also integrable at $0+$ we may use the result that if $G(x;\kappa)$ denotes the distribution function of the $S(\kappa,1)$ law then (see Feller (1971))

$$e^{x^{-\kappa}}G(x;\kappa) \to 0 \quad \text{for} \quad x \downarrow 0. \tag{2.3}$$

Hence, for any $\rho \in \mathbf{R}$, we have

$$\begin{aligned}\int_{\varepsilon} x^{\rho} p(x;\kappa,1)\mathrm{d}x &= \int_{\varepsilon} x^{\rho}\mathrm{d}G(x;\kappa) \\ &= x^{\rho}G(x;\kappa)|_{\varepsilon} + \rho\int_{\varepsilon} x^{\rho-1}G(x;\kappa)\mathrm{d}x,\end{aligned}$$

and the integrability of $x^{\nu+\kappa}p(x;\kappa,\delta,\gamma)$ at $0+$ follows.

We denote by MS (*modified stable*) the class of distributions on the positive halfline whose densities are of the form $p(x;\kappa,\nu,\delta,\gamma)$. Correspondingly, the law determined by $p(x;\kappa,\nu,\delta,\gamma)$ is denoted $MS(\kappa,\nu,\gamma,\delta)$. The subclass of the family of MS laws obtained for $\kappa = \frac{1}{2}$ is the class of GIG (generalised inverse Gaussian) distributions.

Correspondingly, and in analogy with the construction of the generalised hyperbolic distributions, we now introduce the class of *normal modified stable* (NMS) laws. A random variable x is said to be distributed according to the normal modified stable law $NMS(\kappa,\nu,\gamma,\beta,\mu,\delta)$ if it is of the normal variance-mean mixture form

$$x = \mu + \beta\tau + \sqrt{\tau}\varepsilon,$$

with $\varepsilon \sim N(0,1)$ and $\tau \sim MS(\kappa,\nu,\gamma,\delta)$ and τ and ε independent.

We surmise that $MS(\kappa,\nu,\gamma,\delta)$ is infinitely divisible, and in fact selfdecomposable, for arbitrary values of the parameters. However, a general proof of this is not available and, in view of the generally complicated nature of the Lévy density of the GIG laws (cf. Barndorff-Nielsen and Shephard (2001b, Subsection 5.1)), such a proof is likely to be hard to establish. Selfdecomposability does hold when $\kappa = \frac{1}{2}$ (the GIG laws) and also for $\nu = -\kappa$ (the TS laws; see beginning of Section 4). It also holds for arbitrary $\kappa \in (0,1)$ when $-(\nu+\kappa) \in \mathbf{N}$ (the set of natural numbers), cf. Bondesson (1992, Theorem 4.4.1) and, furthermore, for $\kappa = \frac{1}{m}$, $m = 2,3,....$ provided that $\gamma > 0$ and $\nu + \kappa \in \mathbf{N}$. The latter conclusion follows from more recent work of Bondesson (1999) combined with results from Bondesson (1992). In fact, it is shown in Bondesson (1999) that for $\kappa = \frac{1}{n}$ the stable law $S(\kappa,\delta)$ is of type HCM (Hyperbolically Completely Monotonic) and this implies, by the equivalence of *i)* and *v)* in Bondesson (1992, p. 81) that $MS(\frac{1}{n},\nu,\gamma,\delta)$ is GGC (a Generalised Gamma Convolution) and hence selfdecomposable, see Theorem 3.1.1 and the sentence just preceding it in Bondesson (1992).

Provided $MS(\kappa,\nu,\gamma,\delta)$ is infinitely divisible the same is true of $NMS(\kappa,\nu,\gamma,\beta,\mu,\delta)$. In this case we say that a stochastic process y^* is an NMS Lévy process if it is a Lévy process with $y^*(1)$ following the $NMS(\kappa,\nu,\gamma,\beta,\mu,\delta)$ law. We may then, alternatively, characterise y^* as being the sum of a drift term μt and the subordination of Brownian motion with drift b^β by the $MS(\kappa,\nu,\gamma,\delta)$ subordinator, i.e. the subordinator τ^* such that $\tau^*(1)$ is distributed as $MS(\kappa,\nu,\gamma,\delta)$. The class of $NGIG$ (or GH) Lévy processes are recovered by letting $\kappa=\frac{1}{2}$.

3. Subordination

In this Section, for use below, we list a few basic facts on subordination.

A *subordinator* is a Lévy process τ such that $\tau(t)\in(0,\infty)$ for all $t>0$. We assume throughout that τ has no drift, i.e. τ is a pure jump process, and that its life time is infinite in the sense that $\inf\{t\geq 0:\tau(t)=\infty\}=\infty$. The cumulant function of $\tau(t)$ is $tk(\theta)$ where

$$\kappa(\theta)=a_0\theta-\int_0^\infty(1-e^{-\theta x})V(\mathrm{d}x),$$

with $a_0\geq 0$ and where V is the Lévy measure of $\tau(1)$.

Consider a filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_{[0,\infty)},P)$, suppose that on this space there is defined a Lévy process x and a subordinator τ, with x and z independent, and let $y=x\circ\tau$ be the subordination of x by τ, i.e. $y(t)=x(\tau(t))$. Then y is also a Lévy process termed the *subordination of x by τ*, and x is called the *subordinand*. In this case,

$$\psi(\zeta)=\kappa(-\phi(\zeta))$$

where ϕ and ψ denote the characteristic functions of $x(1)$ and $y(1)$, respectively. If the characteristic triplet of x is denoted (a,b,U) then y has characteristic triplet $(a^\sharp,b^\sharp,U^\sharp)$ given by

$$a^\sharp=a_0a+\int_{\mathbf{R}_+}\int_{|x|\leq 1}xP(\mathrm{d}x\ddagger x(t))V(\mathrm{d}t) \tag{3.1}$$

$$b^\sharp=a_0b \tag{3.2}$$

and

$$U^\sharp(\mathrm{d}x)=a_0U(\mathrm{d}x)+\int_{\mathbf{R}_+}P(\mathrm{d}x\ddagger x(t))V(\mathrm{d}t). \tag{3.3}$$

When the probability and Lévy densities exist, formula (3.3) takes the form

$$u^\sharp(x)=a_0u(x)+\int_{\mathbf{R}_+}p(x;t)v(t)\mathrm{d}t \tag{3.4}$$

where $p(x;t)$ is the density of the law of $x(t)$. If, moreover, $a_0=0$ then (3.1) may be written as

$$a^\sharp=\int_{|x|\leq 1}\mathrm{sign}x\,u^\sharp(x)\mathrm{d}x. \tag{3.5}$$

In particular, if x is Brownian motion with drift b^β then $\mathrm{C}\{\zeta \ddagger x(1)\} = -\frac{1}{2}\zeta^2 + i\beta\zeta$ and setting

$$y^*(t) = \mu t + b^\beta(\tau(t)) \tag{3.6}$$

we find that

$$\mathrm{C}\{\zeta \ddagger y^*(1)\} = \mu t + t\bar{\mathrm{K}}\left\{\frac{1}{2}\zeta^2 - i\beta\zeta \ddagger \tau(1)\right\} \tag{3.7}$$

$$\begin{aligned} p(x;t) &= \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}(x-\beta t)^2/t} \\ &= \frac{1}{\sqrt{2\pi}} t^{-1/2} e^{\beta x} e^{-\frac{1}{2}(x^2 t^{-1} + \beta^2 t)} \end{aligned} \tag{3.8}$$

and

$$u^\sharp(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}_+} t^{-1/2} e^{-\frac{1}{2}(x^2 t^{-1} + \beta^2 t)} v(t) \mathrm{d}t e^{\beta x}. \tag{3.9}$$

On recalling that the inverse Gaussian law $IG(\delta, \gamma)$ has probability density function

$$p_{IG}(x;\delta,\gamma) = \frac{\delta}{\sqrt{2\pi}} e^{\delta\gamma} x^{-3/2} e^{-\frac{1}{2}(\delta^2 x^{-1} + \gamma^2 x)} \tag{3.10}$$

one observes that the latter formula may be rewritten in the easily memorised form

$$\bar{u}^\sharp(x) = \int_{\mathbf{R}_+} p_{IG}(t;x;\beta)\bar{v}(t)\mathrm{d}t, \tag{3.11}$$

where $\bar{u}^\sharp(x) = xu^\sharp(x)$ and $\bar{v}(t) = tv(t)$.

4. TS and NTS laws

The Lévy density of $TS(\kappa, \delta, \gamma)$ is

$$u(x) = u(x;\kappa,\delta,\gamma) = \delta 2^\kappa \frac{\kappa}{\Gamma(1-\kappa)} x^{-1-\kappa} e^{-\frac{1}{2}\gamma^{1/\kappa} x} \tag{4.1}$$

and if a random variable x follows the $TS(\kappa, \delta, \gamma)$ law then x has cumulant transform

$$\log \mathrm{E} e^{-\theta x} = \delta\gamma - \delta(\gamma^{1/\kappa} + 2\theta)^\kappa, \tag{4.2}$$

with its expectation and variance being

$$2\kappa\delta\gamma^{(\kappa-1)/\kappa} \quad \text{and} \quad 4\kappa(1-\kappa)\delta\gamma^{(\kappa-2)/\kappa}. \tag{4.3}$$

The fact that the distribution $TS(\kappa, \delta, \gamma)$ is selfdecomposable, which is important in connection with the discussion in Sections 7 and 9 below, follows immediately from (4.1).

For general $\kappa \in (0,1)$, explicit expressions of $p(x;\kappa,\delta)$, and hence of $p(x;\kappa,\delta,\gamma)$, are known only in the form of series representations. Specifically we have (cf. for instance Feller (1971, p. 583))

$$p(t;\kappa,\delta) = \frac{1}{2\pi}\delta^{-1/\kappa} \sum_{k=1}^{\infty} (-1)^{k-1} \sin(k\pi\kappa) \frac{\Gamma(k\kappa+1)}{k!} 2^{k\kappa+1} (t/\delta^{1/\kappa})^{-k\kappa-1}. \tag{4.4}$$

However, the two cases $\kappa = \frac{1}{2}$ and $\kappa = \frac{1}{3}$ are exceptional in this respect.

Example 4.1 *$IG(\delta, \gamma)$ law   With $\kappa = \frac{1}{2}$ we have $TS(\frac{1}{2}, \delta, \gamma) = IG(\delta, \gamma)$ where $IG(\delta, \gamma)$ denotes the inverse Gaussian distribution with probability density function (3.10).* $\square$

Example 4.2 *$TS(\frac{1}{3}, \delta, \gamma)$ law   It is known (see Roberts and Kaufman (1966, p. 79)) that the Laplace transform of the function*

$$x^{-2/3} K_{\frac{1}{3}}(2x^{-1/2})$$

(where $K_{\frac{1}{3}}$ is a Bessel function) is given by

$$\frac{\pi}{\sqrt{3}} e^{-3\theta^{1/3}}.$$

From this we obtain that the probability density function of the $TS(\frac{1}{3}, \delta, \gamma)$ law is[2]

$$\frac{\sqrt{2}}{\pi} \delta^{3/2} e^{\delta^{3/2}\gamma} x^{-3/2} K_{\frac{1}{3}}((\tfrac{2}{3}\delta)^{3/2} x^{-1/2}) e^{-\frac{1}{2}\gamma^3 x}. \tag{4.5}$$

$\square$

Next, let x denote a random variable of the form $x \overset{\mathcal{L}}{=} \mu + \beta\tau + \sqrt{\tau}\varepsilon$ where $\tau \sim TS(\kappa, \delta, \gamma)$, $\varepsilon \sim N(0, 1)$ and τ and ε are independent. The notation $\overset{\mathcal{L}}{=}$ denotes the two variables are equal in law. We then say that x follows the *normal tempered stable law $NTS(\kappa, \gamma, \beta, \mu, \delta)$*[3].

The probability density function of $NTS(\kappa, \gamma, \beta, \mu, \delta)$ has thus a mixture representation which, in the above notation and assuming for simplicity that the location parameter μ is 0, may be written as

$$\begin{aligned} p(x; \kappa, \gamma, \beta, 0, \delta) &= \frac{1}{\sqrt{2\pi}} e^{\delta\gamma} e^{\beta x} \int_0^\infty t^{-1/2} e^{-\frac{1}{2}(x^2 t^{-1} + \beta^2 t)} p(t; \kappa, \delta, \gamma) \mathrm{d}t \\ &= \frac{1}{\sqrt{2\pi}} e^{\delta\gamma} e^{\beta x} \int_0^\infty t^{-1/2} e^{-\frac{1}{2}(x^2 t^{-1} + \alpha^2 t)} p(t; \kappa, \delta) \mathrm{d}t, \end{aligned} \tag{4.6}$$

where $\alpha = \sqrt{\beta^2 + \gamma^{1/\kappa}}$. Transforming the latter integral by the substitution $s = t^{-1}$ we obtain

$$p(x; \kappa, \gamma, \beta, 0, \delta) = \frac{1}{\sqrt{2\pi}} e^{\delta\gamma} e^{\beta x} \int_0^\infty s^{-3/2} e^{-\frac{1}{2}(x^2 s + \alpha^2 s^{-1})} p(s^{-1}; \kappa, \delta) \mathrm{d}s. \tag{4.7}$$

Example 4.3 *$NTS(\frac{1}{3}, \alpha, \beta, 0, 1)$ law.   From (4.5) and (4.7) we find that the probability density of $NTS(\frac{1}{3}, \alpha, \beta, 0, 1)$ is*

$$p(\frac{1}{3}, \alpha, \beta, 0, 1) = (\delta/\pi)^{3/2} e^{\delta\gamma} e^{\beta x} \int_0^\infty e^{-\frac{1}{2}(\alpha^2 s^{-1} + x^2 s)} K_{\frac{1}{3}}((\tfrac{2}{3}\delta)^{3/2} s^{1/2}) \mathrm{d}s.$$

[2]We learned this result from Preben Blæsild (private communication). For $\gamma = 0$ the result specialises to a formula for $S\left(\frac{1}{3}, \delta\right)$ given in Uchaikin and Zolotarev (1999, p. 106).

[3]In particular, for $\kappa = \frac{1}{2}$ we have that $NTS(\frac{1}{2}, \gamma, \beta, \mu, \delta)$ is the same as the normal inverse Gaussian law $NIG(\alpha, \beta, \mu, \delta)$ with $\alpha = \sqrt{\beta^2 + \gamma^2}$.

A reduction of the integral does not seem possible. □

The Lévy density of the $NTS(\kappa,\gamma,\beta,\mu,\delta)$ law is derived in the next Section.

We note, furthermore, that the NTS laws are selfdecomposable, as follows from a result of Sato (2001b), Sato (2001a).

The limiting behaviour of $p(x;\kappa,\gamma,\beta,0,\delta)$ for $x \to \pm$ is determined by the limiting behaviour of $p(x;\kappa,\delta)$ for $x \to \infty$. On account of the series representation (4.4) we have

$$p(x;\kappa,\delta) \sim \delta 2^{\kappa} \frac{\Gamma(1+\kappa)}{\Gamma(\kappa)\Gamma(1-\kappa)} x^{-\kappa-1},$$

for $x \to \infty$. It follows, using also the fact that the density function (A.1) of the GIG law integrates to 1, that the integral in (4.7) behaves asymptotically as

$$\int_0^\infty s^{\kappa-1/2} e^{-\frac{1}{2}(x^2 s+\alpha^2 s^{-1})} \mathrm{d}s = \delta 2^{\kappa+1} \frac{\Gamma(1+\kappa)}{\Gamma(\kappa)\Gamma(1-\kappa)} \alpha^{\kappa+\frac{1}{2}} |x|^{-\kappa-\frac{1}{2}} K_{\kappa+\frac{1}{2}}(\alpha|x|).$$

Since, for $x \to \infty$,

$$K_\lambda(x) \sim \sqrt{\frac{\pi}{2}} x^{-1/2} e^{-x} \tag{4.8}$$

we finally get

$$p(x;\kappa,\gamma,\beta,0,\delta) \sim 2^{\kappa+1}\delta e^{\delta\gamma} \frac{\Gamma(1+\kappa)}{\Gamma(\kappa)\Gamma(1-\kappa)} \alpha^{\kappa+\frac{1}{2}} |x|^{-\kappa-1} e^{-\alpha|x|+\beta x}, \tag{4.9}$$

as $x \to \pm\infty$. In particular, then, the NTS laws have semiheavy tails.

5. TS and NTS Lévy processes

In the notation of Sections 3 and 4, let τ be a TS Lévy process, i.e. a Lévy process such that for some values of κ,δ,γ we have $\tau(1) \sim TS(\kappa,\delta,\gamma)$. From (4.1) we find that for any $t > 0$

$$tu(x;\kappa,\delta,\gamma) = u(x;\kappa,t\delta,\gamma) \tag{5.1}$$

and this implies that

$$\tau(t) \sim TS(\kappa,t\delta,\gamma)$$

for all $t > 0$.

Next, let y^* be the subordination by τ of Brownian motion with drift β, denoted b^β, plus a drift term μt. In other words, y^* is of the form $y^*(t) = \mu t + b^\beta(\tau(t))$. We then say that y^* is a normal tempered stable Lévy process or an NTS Lévy process. Combining (3.11), (5.1) and (4.2) we find that

$$y^*(t) \sim NTS(\kappa,\gamma,\beta,t\mu,t\delta)$$

and

$$\log \mathrm{E}\{\exp(\theta y^*(t))\} = tk(\theta)$$

where

$$k(\theta) = \mu\theta + \delta\gamma - \delta(\alpha^2 - (\beta + \theta)^2)^\kappa \tag{5.2}$$

and $\alpha = \sqrt{\beta^2 + \gamma^{1/\kappa}}$. Thus, in particular, all the one-dimensional laws of y^* are normal tempered stable, with the same parameters κ, γ, β while the location-scale parameters μ and δ are both proportional to t. This is an important property, well known for the special case of $\kappa = \frac{1}{2}$, i.e. for the NIG Lévy processes.

Furthermore, using the formulae (3.9) and (4.1) it follows that the Lévy density of $y^*(1)$ is

$$u^\sharp(x) = \frac{\delta}{\sqrt{2\pi}} \frac{\kappa 2^{\kappa+1}}{\Gamma(1-\kappa)} \alpha^{\kappa+\frac{1}{2}} |x|^{-(\kappa+\frac{1}{2})} K_{\kappa+\frac{1}{2}}(\alpha|x|) e^{\beta x}. \tag{5.3}$$

For $x \to 0$ we find, by (A.2),

$$u^\sharp(x) \sim \frac{\delta}{\sqrt{2\pi}} \kappa 2^{2\kappa+\frac{1}{2}} \frac{\Gamma(\kappa + \frac{1}{2})}{\Gamma(1-\kappa)} |x|^{-2\kappa-1} \tag{5.4}$$

while for $|x| \to \infty$

$$u^\sharp(x) \sim \frac{\delta}{2} \frac{\kappa 2^{\kappa+1}}{\Gamma(1-\kappa)} \alpha^\kappa |x|^{-1-\kappa} e^{-\alpha|x|+\beta x}, \tag{5.5}$$

where we have used (4.8).

6. Relation to NG and $RLPE$ Lévy processes

In this Section we discuss the relations of the NMS Lévy processes to the NG (*Normal Gamma*) and the $RLPE$ Lévy processes (*Regular Lévy Processes of Exponential type*). The former class includes the variance gamma Lévy processes studied by Madan et al (see Madan and Seneta (1990), Madan, Carr, and Chang (1998)). The latter class has been also referred to as Generalised Truncated Levy Processes or, in Carr, Geman, Madan, and Yor (2001), as CGMY processes; see further in subsection 6.2 below.

6.1. NG Lévy processes

The NG (normal gamma) Lévy processes are obtained, like the NMS Lévy processes, by subordination of Brownian motion with drift β, the subordinator $\tau^*(t)$ being now a gamma Lévy process. The special case $\beta = 0$ gives the variance gamma Lévy process.

Let

$$\frac{(\psi^2/2)^\nu}{\Gamma(\nu)} x^{\nu-1} e^{-\frac{1}{2}\psi^2 x},$$

be the probability density function of $\tau^*(1)$ and let $y^*(t)$ denote the resulting NG (normal gamma) Lévy process. Then the log Laplace transform of $y^*(1)$ is

$$\mu\theta + \nu \log\{1 + (\beta/\psi)^2 - (\beta/\psi + \theta/\psi)^2\}.$$

The point we wish to note here is that this occurs as the limit for $\kappa \downarrow 0$ of the log Laplace transform (5.2) provided in the latter we have β fixed whereas δ and γ are chosen as functions of κ such that $\kappa\delta\gamma = \nu$ and $\gamma = \psi^{2\kappa}$. In other words, for $\kappa \downarrow 0$ the $NTS(\kappa, \psi^{2\kappa}, \beta, 0, \nu/\kappa\psi^{2\kappa})$ Lévy process converges in law to the $NG(\nu, \beta, \psi)$ Lévy process.

6.2. $RLPE$ Lévy processes

The asymptotic relations (5.4) and (5.5) show that the NTS Lévy processes belong to the class of $RLPE$ processes (*Regular Lévy Processes of Exponential type*), as defined in Barndorff-Nielsen and Levendorskii (2001). Another subclass of $RLPE$ consists of the Lévy processes z for which $z(1)$ has Lévy density of the form

$$u(x) = \begin{array}{ll} C_-|x|^{-1-A}e^{-B_-|x|} & \text{for} \quad x < 0 \\ C_+x^{-1-A}e^{-B_+x} & \text{for} \quad x > 0. \end{array} \tag{6.1}$$

Such processes (or subclasses thereof) have been considered by Novikov (1994), Koponen (1995), Mantegna and Stanley (2000), Boyarchenko and Levendorskii (1999), Boyarchenko and Levendorskii (2000a), Boyarchenko and Levendorskii (2000b), Boyarchenko and Levendorskii (2000c), Boyarchenko and Levendorskii (2000d), Carr, Geman, Madan, and Yor (2001), and Rosinski (2001).

Comparing to the formulae (5.3), (5.4) and (5.5) one sees that the NTS Lévy processes and the extended Koponen class to a large extent behave similarly at 0 and at $\pm\infty$, provided $C_- = C_+$ and $0 < A < 1$. Note, however, that the NTS processes have smooth probability densities and smooth Lévy densities. Also, the NTS processes are representable by subordination of Brownian motion with drift. To what extent this is the case for the extended Koponen class appears to be an open question.

7. MS-OU and OU-MS processes

7.1. The basic models

In recent work Barndorff-Nielsen and Shephard (2001b) and Barndorff-Nielsen and Shephard (2001c) have developed non-negative OU processes as a building block for a new type of stochastic volatility (SV) model. These SV models will be discussed in the next section, but for the moment we focus on the OU processes. They are given by the solution to the stochastic differential equation (SDE)

$$\mathrm{d}\sigma^2(t) = -\lambda\sigma^2 \mathrm{d}t + \mathrm{d}x(\lambda t).$$

Here the rate parameter λ is arbitrary positive and $x(t)$ is a non-negative homogeneous background driving Lévy process (BDLP) — that is it is a process with independent, stationary and non-negative increments. This means that the BDLP is a subordinator (see, for example, Bertoin (1996) and Sato (1999)). The OU process is representable in law as

$$\sigma^2(t) = e^{-\lambda t}\sigma^2(0) + e^{-\lambda t}\int_0^t e^{\lambda s}\mathrm{d}x(\lambda s).$$

It is possible to construct an OU process with a marginal distribution which is a given MS distribution, irrespectively of the value of λ, if and only if the MS distribution is self-decomposable. We call such a process a MS-OU process. We have discussed this criteria above, and from now we will assume it holds. It certainly holds for the important TS case. An alternative modelling approach is to assume that $x(t)$ is a MS Lévy process which, of course, requires that the MS law is infinitely divisible (plus an additional minor regularity condition, see Wolfe (1982)). We then call the corresponding OU model an OU-MS process.

In the OU-TS case the Lévy density of the BDLP is

$$u(x) = \delta 2^{\kappa}\frac{\kappa}{\Gamma(1-\kappa)}x^{-1-\kappa}\exp\left(-\frac{1}{2}\gamma^{1/\kappa}x\right), \quad \kappa \in (0,1). \tag{7.1}$$

The corresponding Lévy density for the BDLP of the TS-OU process is, using the general theory of OU processes (Barndorff-Nielsen and Shephard (2001b),

$$\begin{aligned}\widetilde{u}(x) &= -u(x) - xu'(x)\\ &= \delta 2^{\kappa}\frac{\kappa}{\Gamma(1-\kappa)}\left(\kappa x^{-1} + \frac{1}{2}\gamma^{1/\kappa}\right)x^{-\kappa}\exp\left(-\frac{1}{2}\gamma^{1/\kappa}x\right).\end{aligned}$$

This shows that the BDLP of the TS-OU is the sum of a TS Lévy process with Lévy density

$$\delta 2^{\kappa}\frac{\kappa^2}{\Gamma(1-\kappa)}x^{-1-\kappa}\exp\left(-\frac{1}{2}\gamma^{1/\kappa}x\right),$$

plus a compound Poisson process[4]. The Lévy density of the compound Poisson process is

$$\delta 2^{\kappa-1}\frac{\kappa}{\Gamma(1-\kappa)}\gamma^{1/\kappa}x^{-\kappa}\exp\left(-\frac{1}{2}\gamma^{1/\kappa}x\right)$$

[4]Recall in general, if z is a nonnegative Lévy process with

$$\bar{\mathrm{K}}\{\theta \ddagger z(1)\} = -\int_0^\infty (1-e^{-\theta x})U(\mathrm{d}x)$$

and if

$$c = U(0,\infty) < \infty$$

then z is a compound Poisson process, the independent summands of which have distribution function $F(x) = c^{-1}U(0,x]$ while the Poisson process has rate c.

which we rewrite as

$$\delta\gamma\kappa\frac{(\gamma^{1/\kappa}/2)^{1-\kappa}}{\Gamma(1-\kappa)}x^{-\kappa}\exp\left(-\frac{1}{2}\gamma^{1/\kappa}x\right)$$

showing that the summands of the process are $\Gamma(1-\kappa,\frac{1}{2}\gamma^{1/\kappa})$-distributed while the Poisson process has rate $\delta\gamma\kappa$.

8. Series representation and simulation

8.1. Rosinski's method

We require to simulate from objects such as

$$e^{-\lambda t}\int_0^t e^{\lambda s}\mathrm{d}x(\lambda s) \overset{\mathcal{L}}{=} e^{-\lambda t}\int_0^{\lambda t} e^{s}\mathrm{d}x(s).$$

In some recent work Rosinski (2001) has shown how to simulate from stochastic integrals of the type

$$\int_0^t f(s)\mathrm{d}x(s)$$

when the Lévy density is of the form

$$u(x) = Ax^{-\alpha-1}e^{-Bx}, \quad x > 0.$$

This covers the TS-Lévy process and so can be used to sample the innovations for the TS-OU and OU-TS processes. In particular

$$e^{-\lambda t}\int_0^{\lambda t} e^{s}\mathrm{d}x(s) \overset{\mathcal{L}}{=} \sum_{i=1}^{\infty}\exp\left(-\lambda t r_i\right)\min\left\{\left(\frac{a_i\kappa}{A\lambda t}\right)^{-1/\kappa}, e_i v_i^{1/\kappa}\right\}, \tag{8.1}$$

where

$$\{e_i\}, \{v_i\}, \{a_i\}, \{r_i\}$$

are independent of one another and over i except for the $\{a_i\}$ process. Here the $\{e_i\}$ are exponential with mean $1/B$, $\{v_i\}$ and $\{r_i\}$ are i.i.d. standard uniforms. Further the $a_1 < ... < a_i < ...$ are the arrival times of a Poisson process with intensity 1.

In the $TS(\kappa,\delta,\gamma)$-OU case we have that

$$A = \delta 2^{\kappa}\frac{\kappa^2}{\Gamma(1-\kappa)}, \quad B = \frac{1}{2}\gamma^{1/\kappa},$$

which allows us to simulate the infinite activity part of the problem using (8.1), while the additional compound Poisson process component of the innovation equals in law

$$\sum_{i=1}^{N(\lambda t)}\exp\left(-\lambda t r_i^*\right)c_i, \quad \text{where} \quad c_i \overset{i.i.d.}{\sim} \Gamma\left(1-\kappa, \frac{1}{2}\gamma^{1/\kappa}\right)$$

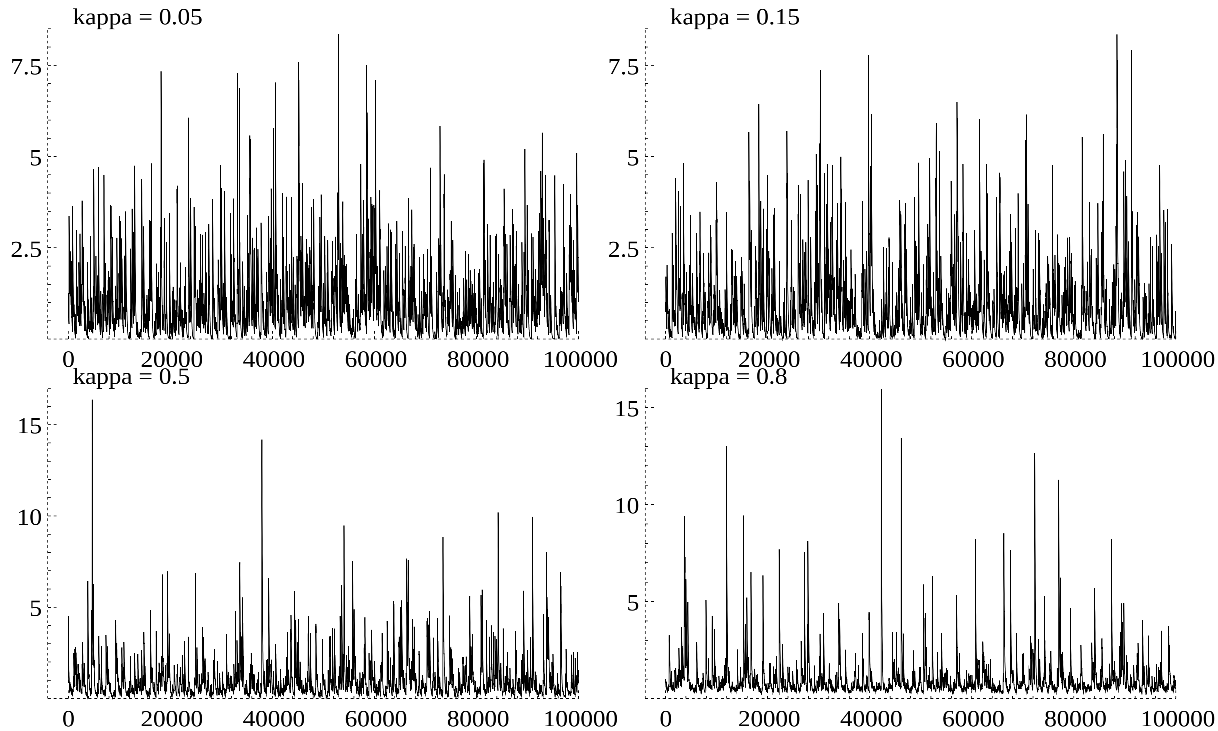

Figure 8.1: *Graph of $\sigma^2(t)$ process against time TS(κ, δ, γ)-OU process for a variety of values of τ. For each value of κ the parameters δ and γ are choosen to that the marginal distribution of the process, TS(κ, δ, γ), has mean and variance of one.*

where the $\{r_i^*\}$ are i.i.d. standard uniforms while $N(1)$ has expected value $\delta\gamma\kappa$. To illustrate these results we have drawn in Figure 8.1 sample paths from the $TS(\kappa, \delta, \gamma)$-OU process for a variety of values of κ. For each selected κ we have chosen δ and γ so that the unconditional distribution of $\sigma^2(t)$ has a mean and variance of one. This is ensured by using the expressions (4.3). We can see from the graph that as κ increases so the tail of the marginal distribution lengthens.

Further, Rosinski (2001) shows that if we select A, B, α to fit the Lévy density of the $TS(\kappa, \delta, \gamma)$, which is (7.1), then

$$\sigma^2(0) \stackrel{\mathcal{L}}{=} \sum_{i=1}^{\infty} \min\left\{ \left(\frac{a_i \kappa}{A}\right)^{-1/\kappa}, e_i v_i^{1/\kappa} \right\}. \tag{8.2}$$

Here

$$A = \delta 2^{\kappa} \frac{\kappa}{\Gamma(1-\kappa)}, \quad B = \frac{1}{2}\gamma^{1/\kappa}.$$

An example of how the infinite sum (8.2) behaves is given in Figure 8.2 which graphs the logarithm of the individual terms, against the value of the index i. It shows that for each value of κ the series is dominated by the first few terms, although as κ goes to one this becomes less sharp.

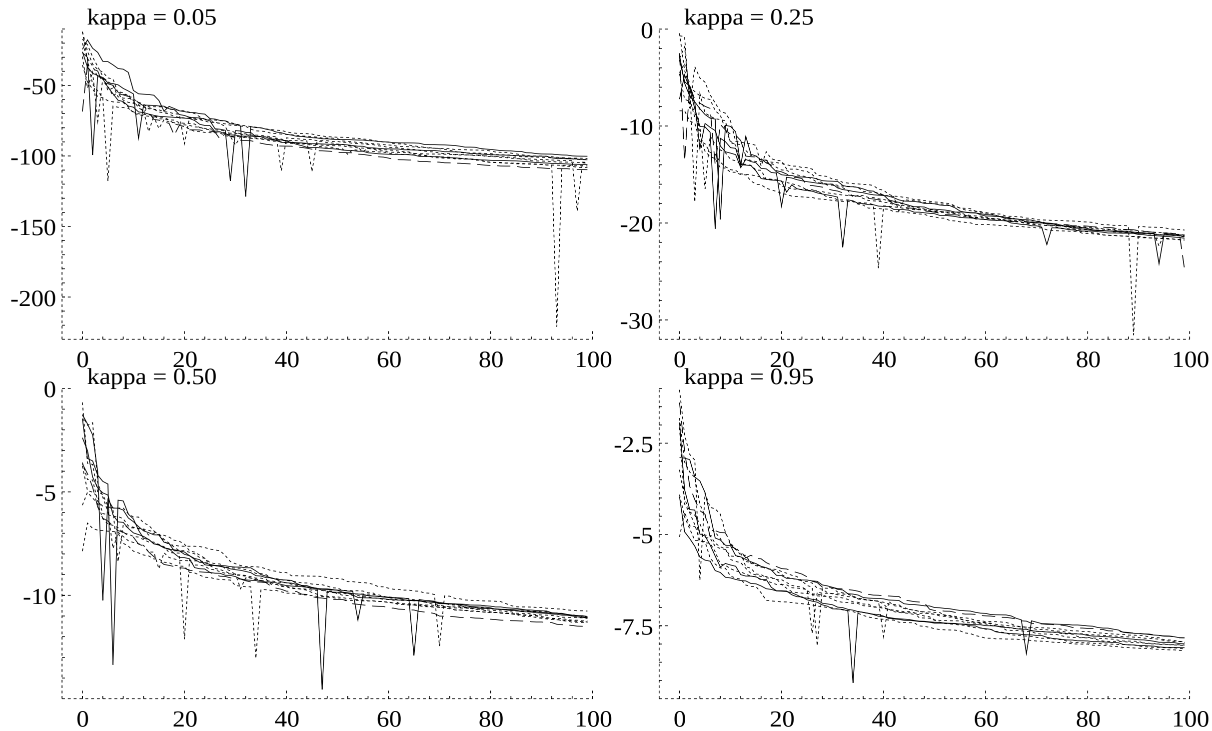

Figure 8.2: *Graph of 5 simulations of the log of the individual terms* $\min\left\{\left(\frac{a_i\kappa}{A}\right)^{-1/\kappa}, e_i v_i^{1/\kappa}\right\}$ *out of the infinite series (8.2) for a TS(*$\kappa, 0.5, 1$*) problem. This shows the rate of decay of these terms.*

8.2. Estimating the normal tilted stable density

The above algorithm for sampling from $\sigma^2(0)$ gives us a general algorithm for sampling from a $TS(\kappa, \delta, \gamma)$ variables, and so from the corresponding normal tilted stable. To gain understanding of the latter variable we have drawn an estimated version of its density in Figure 8.3 for various values of κ. Throughout we choose δ and γ such that the mean of the TS is 0.6^2 and its variance is fixed at 0.05, 0.2, 1 or 5 for Figures (a), (b), (c) and (d) respectively. The estimator is based upon

$$f(x) = \frac{1}{M}\sum_{i=1}^{M}\frac{1}{\sqrt{\sigma^{2j}}}\phi(x/\sqrt{\sigma^{2j}}), \quad \text{where} \quad \sigma^{2j} \overset{i.i.d.}{\sim} TS(\kappa, \delta, \gamma), \tag{8.3}$$

and $\phi(x)$ denotes the density of a standard normal variable. Throughout we selected $M = 26,000$ while we have plotted not the estimator but the log of the estimator. This is done in order to focus on the tail behaviour of the $NTS(\kappa, \gamma, 0, 0, \delta)$ distribution.

This picture is really important for it shows two things we already know and one new thing. When κ is close to zero we have a density which is close to being a normal mixed with a gamma. This has linear tails and is typically regarded as being too thin for financial data. When κ is close to being one half, this corresponds to a normal inverse Gaussian density which has been

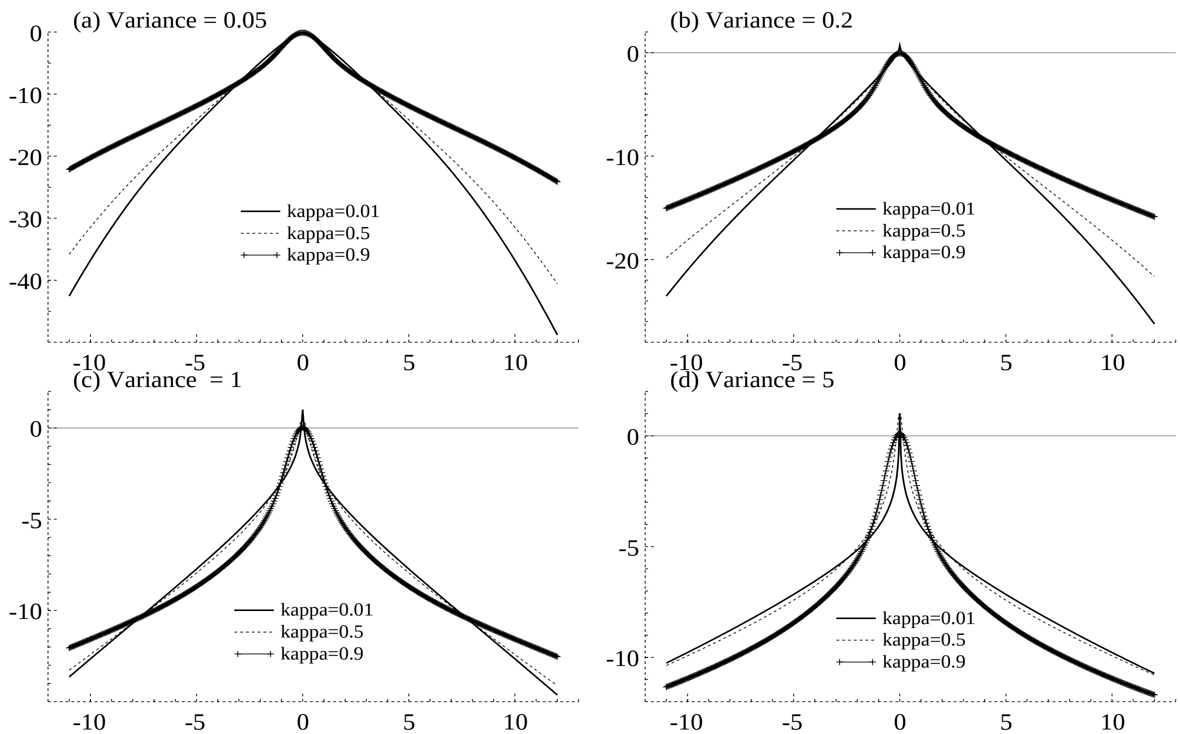

Figure 8.3: *Graph of the estimated log-density of a symmetric NMS variable using a TS(*κ, δ, γ*) model for the scale. Here* κ *is varied, while* δ *and* γ *are correspondingly choosen to ensure the mean of the TS variable is* 0.6^2 *while the variance is 0.05, 0.2, 1 or 5.*

used successfully in a number of financial studies. This has slower tail decay than linear. It is particularly successful at fitting the returns from exchange rate series. When κ is bigger than one half the tails are even heavier. This is important for equity returns typically have heavier tails than exchange rate data and the flexibility to deal with this is very helpful. Of course an alternative to this class, which also has this feature, is the Student t. This is a normal mixed with an inverse gamma. Although this has advantages and is considered in some detail by Barndorff-Nielsen and Shephard (2001b), it has the disadvantage that its associated Lévy density is quite complicated. The NTS alternative is compelling for it has the advantage of the fatter tails while being easy to handle mathematically.

9. Applications to financial economics

9.1. MS-OU and OU-MS based SV models

We can use MS-OU and OU-MS processes as models for the *instantaneous volatility* $\sigma^2(t)$ in a stochastic volatility model of a log-price $y^*(t)$ in financial economics. In these models the log-price is assumed to follow the solution to the SDE,

$$\mathrm{d}y^*(t) = \left\{\mu + \beta\sigma^2(t)\right\}\mathrm{d}t + \sigma^{1/2}(t)\mathrm{d}w(t), \tag{9.1}$$

where $\sigma^2(t)$ is assumed to be stationary and stochastically independent of the standard Brownian motion $w(t)$. It provides a generalisation of the Brownian motion models that have been frequently used in finance and are used to explain the fat tails and serial dependence in equity and exchange rate returns. Further, their structure is often exploited to price the associated derivatives written on these assets (e.g. Hull and White (1987)). Reviews of the literature on this topic are given in Taylor (1994), Shephard (1996) and Ghysels, Harvey, and Renault (1996).

Recalling the definition of returns over an interval of time of length $\Delta > 0$

$$y_n = y^*(\Delta n) - y^*((n-1)\Delta), \qquad n = 1, 2, ... \tag{9.2}$$

this implies that, whatever the model for σ^2, it follows that

$$y_n|\sigma_n^2 \sim N(\mu\Delta + \beta\sigma_n^2, \sigma_n^2). \tag{9.3}$$

Here

$$\sigma_n^2 = \sigma^{2*}(n\Delta) - \sigma^{2*}\{(n-1)\Delta\}, \quad \text{and} \quad \sigma^{2*}(t) = \int_0^t \sigma^2(u)\mathrm{d}u.$$

In econometrics $\sigma^{2*}(t)$ is called *integrated volatility*, while we call σ_n^2 *actual volatility* (see Barndorff-Nielsen and Shephard (2001d)). Integrated volatility is the crucial quantity which drives the behaviour of the returns. In the OU case this is tractable for we can use the result that (see Barndorff-Nielsen and Shephard (2001b))

$$\begin{aligned} \sigma^{2*}(t) &= \lambda^{-1}\left\{z(\lambda t) - \sigma^2(t) + \sigma^2(0)\right\} \\ &= \lambda^{-1}(1 - e^{-\lambda t})\sigma^2(0) + \lambda^{-1}\int_0^t \left\{1 - e^{-\lambda(t-s)}\right\}\mathrm{d}z(\lambda s). \end{aligned} \tag{9.4}$$

That is, integrated volatility is linear in the initial instantaneous volatility and the Lévy increments.

9.2. Simulation of returns from OU-based SV models

We can simulate returns from the SV model with OU-TS volatility by using (9.3), where the sequence $\left\{\sigma_n^2\right\}$ can be produced by sampling $\left\{z(\lambda n\Delta), \sigma^2(n\Delta)\right\}$ and then applying (9.4). In turn $\left\{z(\lambda n\Delta), \sigma^2(n\Delta)\right\}$ can be produced via the recursion

$$\begin{pmatrix} z(\lambda\Delta) \\ \sigma^2(\Delta) \end{pmatrix} \stackrel{\mathcal{L}}{=} \begin{pmatrix} 0 \\ e^{-\lambda\Delta}\sigma^2(0) \end{pmatrix} + \int_0^t \begin{pmatrix} 1 \\ e^{-\lambda(\Delta - s)} \end{pmatrix} \mathrm{d}z(\lambda s)$$

$$\stackrel{\mathcal{L}}{=} \begin{pmatrix} 0 \\ e^{-\lambda\Delta}\sigma^2(0) \end{pmatrix} \sum_{i=1}^{\infty} \begin{pmatrix} 1 \\ \exp(-\lambda t r_i) \end{pmatrix} \min\left\{\left(\frac{a_i\alpha}{A\lambda t}\right)^{-1/\alpha}, e_i v_i^{1/\alpha}\right\}.$$

9.3. Cumulant functions for integrated volatility

9.3.1. General case

We notice that knowledge of the cumulant function of integrated volatility is sufficient to compute the cumulant function of the log-price. Letting $\varepsilon(t;\lambda) = \lambda^{-1}(1-e^{-\lambda t})$ we have, generically,

$$\begin{aligned}
\overline{\mathrm{K}}\{\theta \ddagger \sigma^{2*}(t)\} &= \log \mathrm{E}\{e^{-\theta\sigma^{2*}(t)}\} \\
&= \lambda \int_0^{\varepsilon(t;\lambda)} (1-\lambda r)^{-1} k(\theta r)\mathrm{d}r + \acute{k}(\theta\varepsilon(t;\lambda)). \qquad (9.5)
\end{aligned}$$

where

$$\acute{k}(\theta) = \overline{\mathrm{K}}\{\theta \ddagger \sigma^2(t)\} \quad \text{and} \quad k(\theta) = \overline{\mathrm{K}}\{\theta \ddagger z(1)\}.$$

An important relationship between these two functions is that (cf. Barndorff-Nielsen and Shephard (2001b))

$$k(\theta) = \theta \acute{k}'(\theta). \qquad (9.6)$$

Following the first draft of this paper the tail behaviour of $\sigma^{2*}(t)$ has been studied by Barndorff-Nielsen and Shephard (2001a) who showed that asymptotically the right hand tail of $\sigma^{2*}(t)$ is tempered stable.

The related result of the conditional (on $\sigma^2(0)$) cumulant function for the integrated volatility can be computed using the following expression:

$$\begin{aligned}
\overline{\mathrm{K}}\{\theta \ddagger \sigma^{2*}(t) - \varepsilon(t;\lambda)\sigma^2(0)|\sigma^2(0)\} &= \log \mathrm{E}\{e^{-\theta\sigma^{2*}(t)}|\sigma^2(0)\} - \theta\varepsilon(t;\lambda)\sigma^2(0) \\
&= \lambda \int_0^{\varepsilon(t;\lambda)} (1-\lambda r)^{-1} k(\theta r)\mathrm{d}r. \qquad (9.7)
\end{aligned}$$

This is quite an important expression for knowing this conditional cumulant is sufficient to give us the conditional cumulant function of the log-price. In turn, this is enough to compute the corresponding option prices. This has been carried out in the case of a gamma-OU by Barndorff-Nielsen and Shephard (2001b) and for many other distributions by Nicolato and Venardos (2000) and Tompkins and Hubalek (2000).

In the tempered stable case

$$\acute{k}(\theta) = \delta\gamma - \delta(\gamma^{1/\kappa} + 2\theta)^{\kappa} \quad \text{and} \quad k(\theta) = -\delta\kappa\theta(\gamma^{1/\kappa} + 2\theta)^{\kappa-1}.$$

Rearranging

$$k(\theta) = -\delta\kappa\gamma^{1-1/\kappa}\theta\left(1 + 2\gamma^{-1/\kappa}\theta\right)^{\kappa-1} \quad \text{where} \quad z = 2\gamma^{-1/\kappa}\theta,$$

we find that the integral in (9.7) takes the form

$$\int_0^{\lambda\varepsilon(t;\lambda)} (1-s)^{-1} k(\lambda^{-1}\theta s)\mathrm{d}s = -\delta\kappa\gamma^{1-1/\kappa}\lambda^{-1}\theta \int_0^{\lambda\varepsilon(t;\lambda)} (1-s)^{-1} s(1+as)^{-1+\kappa}\mathrm{d}s \qquad (9.8)$$

where $a = 2\gamma^{-1/\kappa}\lambda^{-1}\theta$. This can be evaluated analytically in a number of cases, in particular for $\kappa = \frac{1}{2}$ and $\frac{1}{3}$, cf. Appendix C. Furthermore, it has been noted and communicated to us by Friedrich Hubalek that for general κ the integral can be expressed in terms of Lerche's Φ function which is given by a Dirichlet series

$$\Phi(x,k,a) = \sum_{n\geq 0} \frac{x^n}{(k+n)^a}.$$

10. Concluding remarks

In this paper we have developed the modified stable laws and the corresponding normal modified stable. A special case of this structure is the tempered stable and normal tempered stable. The TS and NTS distributions can be used as the basis of Lévy processes on the positive half line and on the real line, respectively. As discussed, the same applies for some of the other MS and NMS distributions, and we speculate that it is, in fact, true generally (further analytic and numerical evidence in favour of this conjecture is available in Bondesson (1999)).

The TS (and other modified stable) distributions can be used as the basis of non-Gaussian OU processes and so can be exploited to construct new stochastic volatility models. The TS-OU and OU-TS processes are very tractable, allowing us, in many cases, to compute the conditional cumulant function of the integrated volatility, which means we have a closed form solution to the option pricing problem for these types of processes. This extends previous work by ourselves, and others, for the GIG family.

11. Acknowledgements

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Appendix A: generalised inverse Gaussian distributions

The generalised inverse Gaussian distribution $GIG(\lambda,\delta,\gamma)$ is the distribution on $(0,\infty)$ having probability density function

$$p(x;\lambda,\delta,\gamma) = \frac{(\gamma/\delta)^\lambda}{2K_\lambda(\delta\gamma)} x^{\lambda-1} \exp\{-\frac{1}{2}(\delta^2 x^{-1} + \gamma^2 x)\} \tag{A.1}$$

The parameters λ, γ and δ are such that $\lambda \in \mathbf{R}$ while γ and δ are both nonnegative and not simultaneously 0. Furthermore K_λ is the modified Bessel function of the third kind and with index λ. Letting $\bar{\gamma} = \delta\gamma$, (A.1) may be reexpressed as

$$p(x; \lambda, \delta, \gamma) = \frac{\bar{\gamma}^{\lambda}}{2K_{\lambda}(\bar{\gamma})}\delta^{-2\lambda}x^{\lambda-1}\exp\left\{-\frac{1}{2}(\delta^2 x^{-1} + \bar{\gamma}^2\delta^{-2}x)\right\}$$

from which it appears that δ^2 is a scale parameter while $\bar{\gamma}$ is invariant under scale transformations. For $\bar{\gamma} = \delta\gamma = 0$ the expression (A.1) should be interpreted in the limiting sense, using the well known result that, for $\lambda > 0$ and $x \downarrow 0$,

$$K_{\lambda}(x) \sim \Gamma(\lambda)2^{\lambda-1}x^{-\lambda} \tag{A.2}$$

It follows immediately from the exponential form of the representation (A.1) that if $x \sim GIG(\lambda, \delta, \gamma)$ then

$$\mathrm{E}e^{-\theta x} = \lambda\log\{1 + 2\theta/\gamma^2\}^{1/2} - \log K_{\lambda}(\delta\gamma) + \log K_{\lambda}\left\{\delta\gamma\left(1 + 2\theta/\gamma^2\right)^{1/2}\right\} \tag{A.3}$$

For $\lambda = -\frac{1}{2}$, (A.1) reduces to the probability density of the *inverse Gaussian distribution* $IG(\delta, \gamma)$. Other special cases of the GIG laws are the *reciprocal inverse Gaussian distribution* that corresponds to $\lambda = \frac{1}{2}$ and is denoted $RIG(\delta, \gamma)$, the *gamma distribution* $\Gamma(\nu, \alpha)$ obtained for $\delta = 0$ and with $\nu > 0$, $\lambda = \nu$ and $\alpha = \gamma^2/2$, and the *reciprocal gamma distribution* $R\Gamma(\nu, \alpha)$ which occurs for $\gamma = 0$ and with $\nu > 0$, $\lambda = -\nu$ and $\alpha = \delta^2/2$. Note that if $x \sim IG(\delta, \gamma)$ then $x^{-1} \sim RIG(\gamma, \delta)$, and if $x \sim \Gamma(\nu, \alpha)$ then $x^{-1} \sim R\Gamma(\nu, \alpha)$. For these four distributions the probability densities are:

$IG(\delta, \gamma)$:

$$p(x) = \frac{\delta}{\sqrt{2\pi}}e^{-\delta\gamma}x^{-3/2}\exp\{-(\delta^2 x^{-1} + \gamma^2 x)/2\} \tag{A.4}$$

$RIG(\delta, \gamma)$:

$$p(x) = \frac{\gamma}{\sqrt{2\pi}}e^{-\delta\gamma}x^{-1/2}\exp\{-(\delta^2 x^{-1} + \gamma^2 x)/2\} \tag{A.5}$$

$\Gamma(\nu, \alpha)$:

$$p(x) = \frac{\alpha^{\nu}}{\Gamma(\nu)}x^{\nu-1}e^{-\alpha x} \tag{A.6}$$

$R\Gamma(\nu, \beta)$:

$$p(x) = \frac{\alpha^{\nu}}{\Gamma(\nu)}x^{-\nu-1}e^{-\alpha x^{-1}} \tag{A.7}$$

The formula for the Lévy density of the GIG law is given, for instance, in Barndorff-Nielsen and Shephard (2001b, Section 5.1).

Appendix B: generalised hyperbolic distributions

To define the generalized hyperbolic distributions, suppose u is a random variable with law $GIG(\lambda, \delta, \gamma)$ and let $x = \mu + u\Delta\beta + uy$ where y follows the m-dimensional normal distribution with mean 0 and variance matrix Δ. For parametric identifiability, Δ is assumed to have determinant 1, i.e. $|\Delta| = 1$. The probability density of x is then

$$\begin{aligned} p(x; \lambda, \alpha, \beta, \mu, \delta, \Delta) &= \frac{1}{(2\pi)^{m/2}\sqrt{|\Delta|}} \frac{(\gamma/\delta)^\lambda \alpha^{m/2-\lambda}}{K_\lambda(\delta\gamma)} \\ &\cdot \left(\delta^2 + R\right)^{(\lambda - m/2)/2} K_{\lambda - m/2}\left\{\alpha\left(\delta^2 + R\right)^{1/2}\right\} \\ &\cdot \exp\left(\langle \beta, x - \mu\rangle\right) \qquad \text{(B.1)} \end{aligned}$$

where $\alpha = \{\gamma^2 + \beta\Delta\beta^\top\}^{1/2}$ and

$$R = (x - \mu)^T \Delta^{-1} (x - \mu)$$

This class of distributions is closed under marginalization and conditioning (with respect to subvectors of x).

The class of m-dimensional normal inverse Gaussian distributions is obtained for $\lambda = -\frac{1}{2}$, while the class of *hyperbolic laws* corresponds to $\lambda = (m+1)/2$. The latter laws are characterized by the fact the graphs of their log densities are hyperboloids.

Note also that the m-dimensional Student distributions are special cases of (B.1), obtained for $\lambda < 0$, $\alpha = \beta = \mu = 0$ and $\Delta = I$. In particular, the Cauchy law occurs by further taking $m = 1$, $\lambda = -\frac{1}{2}$ and $\delta = 1$.

For the one-dimensional case $m = 1$ we denote the *generalized hyperbolic distribution* by $GH(\lambda, \alpha, \beta, \mu, \delta)$. In this case the probability density function (B.1) takes the form

$$p(x; \lambda, \alpha, \beta, \mu, \delta) = \frac{\bar{\gamma}^\lambda \bar{\alpha}^{1/2-\lambda}}{\sqrt{2\pi}\delta K_\lambda(\bar{\gamma})} q\left(\frac{x-\mu}{\delta}\right)^{\lambda - 1/2} K_{\lambda - 1/2}\left(\bar{\alpha} q\left(\frac{x-\mu}{\delta}\right)\right) e^{\bar{\beta}(x-\mu)/\delta} \qquad \text{(B.2)}$$

where q denotes the function $q(x) = \sqrt{(1 + x^2)}$ and $\bar{\beta} = \delta\beta$, $\bar{\gamma} = \delta\gamma$ and $\bar{\alpha} = \delta\alpha = \delta\sqrt{(\gamma^2 + \delta^2)}$ are location-scale invariant parameters.

It follows immediately from (B.2) and (4.8) that for $x \to \pm\infty$ we have

$$p(x; \lambda, \alpha, \beta, 0, \delta) \sim const.|x|^{\lambda - 1} \exp(-\alpha|x| + \beta) \qquad \text{(B.3)}$$

Thus all the generalised hyperbolic laws have semiheavy tails.

Taking, for notational simplicity, $\mu = 0$ it is illuminating to distinguish between the cases $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$, rewriting (B.2) in the three cases as[5]:

[5] It is convenient here to introduce the notation $\bar{K}_\lambda(x) = x^\lambda K_\lambda(x)$ for $\lambda > 0$ and $x > 0$. By (5.9) we have $\bar{K}_\lambda(x) \to \Gamma(\lambda)2^{\lambda-1}$ for $x \downarrow 0$ and we therefore define $\bar{K}_\lambda(0) = \Gamma(\lambda)2^{\lambda-1}$.

for $\lambda > 0$ (when we may let $\delta \downarrow 0$):

$$p(x;\lambda,\alpha,\beta,0,\delta) = \frac{\gamma^{2\lambda}\alpha^{1-2\lambda}}{\sqrt{2\pi}\bar{K}_\lambda(\bar{\gamma})}\bar{K}_{\lambda-1/2}\left\{\alpha\sqrt{\delta^2+x^2}\right\}e^{\beta x} \tag{B.4}$$

for $\lambda = 0$:

$$p(x;\lambda,\alpha,\beta,0,\delta) = \frac{1}{2K_0(\bar{\gamma})}\frac{1}{\sqrt{\delta^2+x^2}}e^{-\alpha\sqrt{\delta^2+x^2}+\beta x} \tag{B.5}$$

for $\lambda = -\nu < 0$ (when we may let $\gamma \downarrow 0$):

$$p(x;-\nu,\alpha,\beta,0,\delta) = \frac{1}{\sqrt{2\pi}\delta\bar{K}_\nu(\bar{\gamma})}q\left(\frac{x}{\delta}\right)^{-2\nu-1}\bar{K}_{\nu+1/2}\left\{\bar{\alpha}q\left(\frac{x}{\delta}\right)\right\}e^{\beta x} \tag{B.6}$$

This specializes, when $\lambda = -\frac{1}{2}$, to the probability density function of the *normal inverse Gaussian distribution:*

$$p(x;\alpha,\beta,\mu,\delta) = a(\alpha,\beta,\mu,\delta)q\left(\frac{x-\mu}{\delta}\right)^{-1}K_1\left\{\delta\alpha q\left(\frac{x-\mu}{\delta}\right)\right\}e^{\beta x} \tag{B.7}$$

where $q(x) = \sqrt{(1+x^2)}$ and

$$a(\alpha,\beta,\mu,\delta) = \pi^{-1}\alpha\exp\left\{\delta\sqrt{(\alpha^2-\beta^2)}-\beta\mu\right\} \tag{B.8}$$

For $\lambda = 1$ we have the density of the one-dimensional *hyperbolic law*

$$p(x;\alpha,\beta,\mu,\delta) = \frac{\bar{\gamma}}{2\delta\bar{\alpha}K_1(\bar{\gamma})}\exp\left\{-\bar{\alpha}q\left(\frac{x-\mu}{\delta}\right)+\bar{\beta}\left(\frac{x-\mu}{\delta}\right)\right\} \tag{B.9}$$

The *Laplace distributions* (symmetric and asymmetric) occur as limiting cases of (B.9) for α, β and μ fixed and $\delta \downarrow 0$.

The *Student distributions* and asymmetric versions of these are obtained by letting the mixing law be $GIG(-\nu,\delta,0)$, that is we are mixing $N(\beta u, u)$ with u following the $R\Gamma(\nu,\delta^2/2)$ law. The resulting probability density is obtained by letting $\gamma \downarrow 0$ in formula (B.6):

$$p(x;\beta,\delta) = \frac{1}{\sqrt{2\pi}\delta\Gamma(\nu)2^{\nu-1}}q\left(\frac{x}{\delta}\right)^{-2\nu-1}\bar{K}_{\nu+1/2}\left(\bar{\beta}q\left(\frac{x}{\delta}\right)\right)e^{\beta x}$$

the Student law with degrees of freedom 2ν

$$\frac{\Gamma(\nu+\frac{1}{2})}{\sqrt{2\pi\nu}\Gamma(\nu)}(1+\frac{x^2}{2\nu})^{-\nu-\frac{1}{2}}$$

occurring for $\beta = 0$ and $\delta = \sqrt{2\nu}$.

Furthermore, for $\delta = 0, \mu = 0$ and $\lambda > 0$ we obtain the *normal gamma law* $N\Gamma(\lambda,\alpha,\rho)$ with 2λ degrees of freedom and probability density function

$$p(x;\lambda,\alpha,\rho) = \frac{(1-\rho^2)^\lambda\alpha^{1/2+\lambda}}{\sqrt{2\pi}\Gamma(\lambda)2^{\lambda-1}}|x|^{\lambda-1/2}K_{\lambda-1/2}(\alpha|x|)e^{\rho\alpha x} \tag{B.10}$$

where $\rho = \beta/\alpha \in]-1,1[$. (In the symmetric case, that is for $\rho = 0$, this distribution is also known as the *variance gamma law.*)

Appendix C: Analytic evaluation of some integrals

For the purpose of evaluation of the integral on the right hand side of (9.8) we note first that

$$\begin{aligned}\int (1-x)^{-1}(1+ax)^{\kappa-1}dx &= \frac{1}{1+a}\int \left\{\frac{1}{1-x}+\frac{a}{1+ax}\right\}(1+ax)^{\kappa}dx \\ &= (1+a)^{-1}I(\kappa;a)+\{(1+a)\kappa\}^{-1}(1+ax)^{\kappa}\end{aligned}$$

where

$$I(\kappa;a)=\int (1-x)^{-1}(1+ax)^{\kappa}dx$$

The integral $I(\kappa;a)$ can be evaluated explicitly for a range of values of κ. In particular, using MAPLE, we have found

$$I(\tfrac{1}{2};a)=-2\sqrt{(1+ax)}+2\sqrt{(1+a)}\operatorname{arctanh}\frac{\sqrt{(1+ax)}}{\sqrt{(1+a)}}$$

$$\begin{aligned}I(\tfrac{1}{3};a) &= -3\sqrt[3]{(1+ax)} \\ &\quad +\frac{1}{2}\frac{1+a}{\left(\sqrt[3]{(a+1)}\right)^2}\ln\left(\left(\sqrt[3]{(1+ax)}\right)^2+\sqrt[3]{(1+ax)}\sqrt[3]{(a+1)}+\left(\sqrt[3]{(a+1)}\right)^2\right) \\ &\quad -\frac{1+a}{\left(\sqrt[3]{(a+1)}\right)^2}\ln\left(\sqrt[3]{(1+ax)}-\sqrt[3]{(a+1)}\right) \\ &\quad +\frac{1+a}{\left(\sqrt[3]{(a+1)}\right)^2}\sqrt{3}\arctan\frac{1}{3}\sqrt{3}\left(\frac{2}{\sqrt[3]{(a+1)}}\sqrt[3]{(1+ax)}+1\right)\end{aligned}$$

$$\begin{aligned}I(\tfrac{1}{4};a) &= -4\sqrt[4]{(1+ax)}+\frac{1+a}{\left(\sqrt[4]{(a+1)}\right)^3}\ln\frac{\sqrt[4]{(1+ax)}+\sqrt[4]{(a+1)}}{\sqrt[4]{(1+ax)}-\sqrt[4]{(a+1)}} \\ &\quad +2\frac{1+a}{\left(\sqrt[4]{(a+1)}\right)^3}\arctan\frac{\sqrt[4]{(1+ax)}}{\sqrt[4]{(a+1)}}\end{aligned}$$

$$\begin{aligned}I(\tfrac{2}{3};a) &= -\frac{3}{2}\left(\sqrt[3]{(1+ax)}\right)^2-\frac{1+a}{\sqrt[3]{(a+1)}}\ln\left(\sqrt[3]{(1+ax)}-\sqrt[3]{(a+1)}\right) \\ &\quad +\frac{1}{2}\frac{1+a}{\sqrt[3]{(a+1)}}\ln\left(\left(\sqrt[3]{(1+ax)}\right)^2+\sqrt[3]{(1+ax)}\sqrt[3]{(a+1)}+\left(\sqrt[3]{(a+1)}\right)^2\right) \\ &\quad -(1+a)\frac{\sqrt{3}}{\sqrt[3]{(a+1)}}\arctan\frac{1}{3}\sqrt{3}\left(\frac{2}{\sqrt[3]{(a+1)}}\sqrt[3]{(1+ax)}+1\right)\end{aligned}$$

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