A discrete Clark-Ocone formula

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Abstract

In this article the Malliavin calculus for a discrete finite time line is developed in the same way as it was done by Holden, Lindstrøm, Øksendal and Ubøe. A modification of the definition of the discrete Malliavin derivative allows us to formulate and to prove a discrete Clark-Ocone formula.

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1 Introduction

In this article I develop a version of the Malliavin calculus for a discrete finite timeline as it was done by Holden et al. in [9, 8]. A minor modification of some definitions and their interpretations gives me the possibility to formulate and to prove the Clark-Ocone formula in this discrete setting.

The notation is much inspired by Meyer's toy Fock space as it is found in [15, 14, 16]. This finite calculus has its analogue in the Maassen kernel calculus of quantum stochastics [13, 10]. There the non-causal non-quantum stochastic calculus is contained as a special case [11]. This approach uses the space of square integrable functions of the symmetric space over the Lebesgue space [7]. Fundamental to the discrete calculus is the Wick product of random variables [6]. This allows an easy definition of the Skorohod integral.

There is a discrete but not finite version of the Clark-Ocone formula in an article by Privault and Schoutens [17]. There the timeline is the set of natural numbers. In contrary to that article I have a measure on my discrete finite set.

Most of the basic material in this article can be found in [9] but for the reader's convenience I have included some proofs that are left out in [9, 8].

2 Basic Definitions, Notations, Facts

Almost all in this section is also contained in [9] but there some of the proofs are omitted. So I repeat the material presented in [9] and include all the proofs.

Let be $N \in \mathbb{N}$ and set $\triangle t = \frac{1}{N}$. Then I take the set

$$\Lambda = \{0, \triangle t, \dots, (N-1)\Delta t\}$$

as a discrete version of the finite time line $[0,1]$. As measure μ on Λ I take the uniform counting measure, i.e. for $A \subset \Lambda$ one has $\mu(A) = \frac{|A|}{N}$. The measure algebra is the potential set of Λ. So the triple $(\Lambda, \mathcal{P}(\Lambda), \mu)$ is my discrete version of the Lebesgue space $([0,1], \mathcal{B}, \lambda)$.

Next I introduce the set

$$\Omega = \{\omega | \omega : \Lambda \to \{-1, +1\}\}$$

and think of each ω as a Bernoulli random variable. On $\mathcal{P}(\Omega)$ I take the uniform probability measure P, i.e. for $S \subset \Omega$ one has $P(S) = \frac{|S|}{|\Omega|} = \frac{|S|}{2^N}$. With respect to P I form $L^2(\Omega, P)$ with the inner product

$$\langle X, Y \rangle_{L^2} = \sum_{\omega \in \Omega} X(\omega) Y(\omega) P(\omega).$$

The space $L^2(\Omega, P)$ is my discrete version of the Wiener space.
It is of $\dim L^2(\Omega, P) = 2^N$ since one has a basis of characteristic functions to each atom $\omega \in \Omega$ scaled with the factor $\sqrt{2^N}$.

Remark 2.1 In nonstandard analysis the introduced discrete Lebesgue space and Wiener space are hyperfinite models for their continuous time counterparts. This was shown by Anderson [5] for the Wiener space.

Definition 2.2 *For $A \in \mathcal{P}(\Lambda)$ I define the functions $\chi_A : \Omega \to \mathbb{R}$ by*

$$\chi_A(\omega) = \prod_{s \in A} \omega(s).$$

$\triangleleft$

Lemma 2.3

$$\sum_{\omega \in \Omega} \chi_C(\omega) P(\omega) = \begin{cases} 1 & \textit{if} \quad C = \emptyset \\ 0 & \textit{if} \quad C \neq \emptyset \end{cases}$$

Proof: For $C = \emptyset$ one has $\chi_\emptyset(\omega) = 1$ independent of ω. So I obtain

$$\sum_{\omega \in \Omega} \chi_\emptyset(\omega) P(\omega) = \sum_{\omega \in \Omega} P(\omega) = 1\,.$$

For $C \neq \emptyset$ the function χ_C takes only the values $+1$ or -1 depending on how often ω has value -1 on the set C. Summing over ω it is enough to show that the product $\chi_C(\omega) = \prod_{s \in C} \omega(s)$ is in the half of the cases -1. Suppose that $|C| = n$. Then the product is -1 if an odd number of -1's occurs in $\prod_{s \in C} \omega(s)$ and $+1$ for an even number of -1's. Thus I show that $\sum_k \binom{n}{2k} = \sum_k \binom{n}{2k+1}$:

$$0 = (1 + (-1))^n = \sum_k \binom{n}{k} (-1)^k 1^{n-k} = \sum_k \binom{n}{2k} - \sum_k \binom{n}{2k+1}$$

$\square$

The next proposition and its corollary are also in [9, 8] but there the obvious proofs are omitted.

Proposition 2.4
The set $\{\chi_A\}_{A \in \mathcal{P}(\Lambda)}$ is an orthonormal system in $L^2(\Omega, P)$.

Proof: First note that for $A, B \in \mathcal{P}(\Lambda)$

$$\chi_A(\omega)\chi_B(\omega) = \prod_{s\in A}\omega(s)\prod_{s\in B}\omega(s) = \prod_{s\in A\triangle B}\omega(s) = \chi_{A\triangle B}(\omega).$$

Thus one sees that

$$\langle \chi_A, \chi_B\rangle = \sum_{\omega\in\Omega}\chi_{A\triangle B}(\omega)P(\omega).$$

For $A = B$ one has $\sum_{\omega\in\Omega}\chi_\emptyset(\omega)P(\omega) = 1$ by the preceding lemma. For $A \neq B$ one obtains $\sum_{\omega\in\Omega}\chi_C(\omega)P(\omega) = 0$ with $C = A \triangle B \neq \emptyset$ by the preceding lemma. So the proposition is proved. □

Corollary 2.5 $\{\chi_A\}_{A\in\mathcal{P}(\Lambda)}$ *is a basis for* $L^2(\Omega, P)$.

Proof:

$$\sharp\{\chi_A\}_{A\in\mathcal{P}(\Lambda)} = \sharp\mathcal{P}(\Lambda) = 2^N = \dim L^2(\Omega, P).$$

□

Notation 2.6 As shorthand notation I set

$$\mathcal{P}_n = \{A \in \mathcal{P}(\Lambda) : |A| = n\},\ \mathcal{P} = \mathcal{P}(\Lambda) = \dot{\cup}_n\mathcal{P}_n.$$

◁

Definition 2.7 *For* $X \in L^2(\Omega, P)$ *I call*

$$X = \sum_{A\in\mathcal{P}} X(A)\chi_A = \sum_n\sum_{A\in\mathcal{P}_n} X(A)\chi_A$$

the Walsh decomposition *of* X. ◁

Proposition 2.8 *Let* $X = \sum_{A\in\mathcal{P}} X(A)\chi_A$. *Then* $E[X] = X(\emptyset)$.

Proof:

$$\begin{aligned} E[X] &= \sum_{\omega\in\Omega}\sum_{A\in\mathcal{P}} X(A)\chi_A(\omega)P(\omega) \\ &= \sum_{A\in\mathcal{P}} X(A)\sum_{\omega\in\Omega}\chi_A(\omega)P(\omega) \ = \ X(\emptyset) \end{aligned}$$

where in the last step I have used lemma 2.3. □

Definition 2.9 *Let* $X = \sum_{A\in\mathcal{P}} X(A)\chi_A$ *and* $Y = \sum_{B\in\mathcal{P}} Y(B)\chi_B$ *be random variables. Then the* Wick product $X \diamond Y$ *is defined by*

$$X \diamond Y = \sum_{C\in\mathcal{P}} (\sum_{A\dot{\cup}B} X(A)Y(B))\chi_C .$$

$\triangleleft$

Remark 2.10

(1)

$$\chi_A \diamond \chi_B = \begin{cases} \chi_{A\cup B}\,, & \text{if} \quad A\cap B = \emptyset \\ 0\,, & \text{otherwise} \end{cases}$$

(2) If $A\cap B = \emptyset$ then $\chi_A \diamond \chi_B = \chi_A \cdot \chi_B$.

Lemma 2.11 $(L^2(\Omega, P), +, \diamond)$ *is a commutative ring with unit* $\chi_\emptyset$.

PROOF: straightforward. $\square$

Definition 2.12 (discrete analogue)

- *A* stochastic process *is a family of random variables* $(X_s)_{s\in\Lambda}$, *i.e. a map* $X : \Omega\times\Lambda \to \mathbb{R}$ *such that for each fixed* $s \in \Lambda$ *the map* $X(\,\cdot\,, s)$ *is in* $L^2(\Omega, P)$.
- *The* Brownian motion B *is the random walk*

$$B : \Omega\times\Lambda \to \mathbb{R},\ B(\omega, t) = \sum_{s<t} \omega(s)\sqrt{\triangle t}\,.$$

- *The* white noise W *over* (Λ, μ) *is the map*

$$W : \Omega\times\mathcal{P}(\Lambda) \to \mathbb{R},\ W(\omega, A) = \sum_{s\in A} \frac{\omega(s)}{\sqrt{\triangle t}}\,.$$

For $t \in \Lambda$ *I set* $W_t(\omega) = W(\omega, \{t\}) = \frac{\omega(t)}{\sqrt{\triangle t}}$ *and call this* pointwise white noise.

- *The* forward increment *of* B *is defined by*

$$\triangle B_t = \triangle B(\omega, t) = B(\omega, t + \triangle t) - B(\omega, t) = \omega(t)\sqrt{\triangle t}.$$

Thus the derivative *of the Brownian motion is the pointwise white noise:*

$$\frac{\triangle B_t}{\triangle t} = \frac{\omega(t)\sqrt{\triangle t}}{\triangle t} = \frac{\omega(t)}{\sqrt{\triangle t}} = W_t.$$

- *Let be* $(X_s)_{s\in\Lambda}$ *an adapted stochastic process. Then the* Itô integral *is defined by*

$$\int X dB = \int X_s dB_s = \sum_s X_s \cdot \triangle B_s = \sum_s X_s \cdot W_s \triangle t.$$

$\lhd$

Remark 2.13 In [8] Holden et. al. define $\widehat{W}(\omega, A) = \sum_{s\in A} \omega(s)\sqrt{\triangle t}$ as white noise. This seem to me the wrong way around because pointwise white noise then would not be the derivative of Brownian motion. Also the discrete Skorohod integral could not be expressed as a discrete Lebesgue integral where the integrand is Wick multiplied by pointwise white noise.

Now I will establish a discrete version of the Wiener-Itô decomposition for random variables $X \in L^2(\Omega, P)$. This is done in the same way as in [9]. Actually the Walsh decomposition is nearly the Wiener-Itô decomposition just written in the notation of a discrete Guichardet space. One should compare with [13]. But since I am working in a discrete measure space, in that the diagonals do not have measure zero, I define the symmetric functions to be zero on diagonals. So the similarity to the continuous time theory is better achieved.

Let be $X = \sum_{A\in\mathcal{P}} X(A)\chi_A$ the Walsh decomposition of X. Then I define for $n > 0$ the symmetric function X_n on Λ^n by

$$X_n(t_1, \ldots, t_n) = \begin{cases} (\triangle t^{n/2} n!)^{-1} X(\{t_1, \ldots, t_n\}), \text{ if } t_i \neq t_j \text{ for } i \neq j \\ 0, \text{ otherwise} \end{cases}$$

where $X(\{t_1, \ldots, t_n\})$ is the Walsh component to $A = \{t_1, \ldots, t_n\}$. For $n = 0$ I set $X_0 = X(\emptyset) = E[X]$. One obtains

$$\begin{aligned} X &= \sum_{A\in\mathcal{P}} X(A)\chi_A = \sum_n \sum_{A\in\mathcal{P}_n} X(A)\chi_A \\ &= \sum_n \sum_{\{t_1,\ldots,t_n\}\in\mathcal{P}_n} X(\{t_1, \ldots, t_n\})\omega(t_1) \cdot \ldots \cdot \omega(t_n) \\ &= \sum_n \sum_{\substack{(t_1,\ldots,t_n)\in\Lambda^n \\ t_1<\ldots<t_n}} n! X_n(t_1, \ldots, t_n) \triangle t^{\frac{n}{2}} \omega(t_1) \cdot \ldots \cdot \omega(t_n) \\ &= \sum_n \sum_{(t_1,\ldots,t_n)\in\Lambda^n} X_n(t_1, \ldots, t_n) \triangle B(t_1) \cdots \triangle B(t_n). \end{aligned}$$

The last term is nothing else than the *discrete Wiener-Itô decomposition.*

3 Conditional expectations

The main proposition 3.2 in this section is from [9] where the proof is left out. I include the proof for pedagogical purpose to demonstrate in which way proofs reduce to some combinatorial arguments in this discrete finite setting.

Notation 3.1 For $B \subset \Lambda$ I denote by $\mathcal{F}_B$ the σ-algebra on Ω generated by the random variables $\{\omega(s) : s \in B\}$. $\triangleleft$

For example for each $s \in \Lambda$ one has

$$\mathcal{F}_{\{s\}} = \{\emptyset, \{\omega : \omega(s) = -1\}, \{\omega : \omega(s) = +1\}, \Omega\}.$$

Since this are the atomic σ-algebras one can construct every $\mathcal{F}_B$ out of them:

$$\mathcal{F}_B = \sigma - alg[\{\{\omega : \omega(s) = -1\}, \{\omega : \omega(s) = +1\} | s \in B\}].$$

Proposition 3.2 *Let $X = \sum_{A\subset\Lambda} X(A)\chi_A$ and $\mathcal{F}_B$ be given. Then the conditional expectation of X with respect to $\mathcal{F}_B$ is given by*

$$E[X|\mathcal{F}_B] = \sum_{A\subset B} X(A)\chi_A.$$

Proof: That $\sum_{A\subset B} X(A)\chi_A$ is $\mathcal{F}_B$-measurable is evident. Further I have to prove that, for every $H \in \mathcal{F}_B$,

$$\int_{\omega\in H} E[X|\mathcal{F}_B]dP = \int_{\omega\in H} X dP\,.$$

The left hand side is

$$\begin{aligned}\int_{\omega\in H} E[X|\mathcal{F}_B]dP &= \sum_{\omega\in H}\sum_{A\subset B} X(A)\chi_A(\omega)P(\omega) \\ &= \sum_{A\subset B} X(A)\sum_{\omega\in H}\chi_A(\omega)P(\omega)\end{aligned}$$

and the right hand side is

$$\begin{aligned}\int_{\omega\in H} XdP &= \sum_{\omega\in H}\sum_{A\subset\Lambda} X(A)\chi_A(\omega)P(\omega)\\ &= \sum_{A\subset\Lambda} X(A)\sum_{\omega\in H}\chi_A(\omega)P(\omega)\,.\end{aligned}$$

So it is sufficient to show that for $H\in\mathcal{F}_B$ and for every $A\not\subset B$ one has

$$\sum_{\omega\in H}\chi_A(\omega)P(\omega)=0\,.$$

If $A\not\subset B$ then there exists an $s_0\in A$ with $s_0\notin B$. But this shows that one can divide the set H into two parts

$$H_{s_0}^-=\{\omega\in H:\omega(s_0)=-1\}\ \text{ and }\ H_{s_0}^+=\{\omega\in H:\omega(s_0)=+1\}$$

and $H=H_{s_0}^-\dot\cup H_{s_0}^+$. Furthermore for each $\omega^-\in H_{s_0}^-$ there exists exactly one $\omega^+\in H_{s_0}^+$ such that $\omega^-(s)=\omega^+(s)$ for all $s\in\Lambda\setminus\{s_0\}$. This shows $\sharp H_{s_0}^-=\sharp H_{s_0}^+$ and therefore $\sum_{\omega\in H}\chi_A(\omega)P(\omega)=0$. Thus it is proved that

$$\int_{\omega\in H}XdP=\sum_{A\subset B}X(A)\sum_{\omega\in H}\chi_A(\omega)P(\omega)=\int_{\omega\in H}E[X|\mathcal{F}_B]dP$$

for every $H\in\mathcal{F}_B$. □

The formula shows that the conditional expectation of X with respect to $\mathcal{F}_B$ depends only on those Walsh components χ_A such that $A\subset B$.

The next observation is implicitly contained in the remark that $\chi_A\diamond\chi_B=\chi_A\cdot\chi_B$ if $A\cap B=\emptyset$, but the interpretation now has another flavour.

Proposition 3.3 *Let be $A,B\subset\Lambda$ and $X,Y\in L^2(\Omega,P)$. Assume $A\cap B=\emptyset$ and that X is $\mathcal{F}_A$-measurable and Y is $\mathcal{F}_B$-measurable. Then*

$$X\diamond Y=X\cdot Y\,.$$

Proof: The measurability assumption shows that the Walsh decompositions of X and Y are

$$X=\sum_{C\subset A}X(C)\chi_C\ \text{ and }\ Y=\sum_{D\subset B}X(D)\chi_D.$$

Thus

$$\begin{aligned} X \diamond Y &= \sum_{C,D} \{X(C)Y(D) : C \subset A, D \subset B, C \cap D = \emptyset\} \chi_{C \cup D} \\ &= \sum_{C,D} \{X(C)Y(D) : C \subset A, D \subset B\} \chi_{C \triangle D} \quad = \quad X \cdot Y \end{aligned}$$

and the proposition is proved. □

If I would define a certain measure m on $\mathcal{P}(\Lambda)$ then it would be equivalent to assume $m(A \cap B) = 0$ instead of $A \cap B = \emptyset$.

In the next definition I introduce the σ-algebras that will constitute my discrete filtration. One should notice that the information of the present is not yet available by these algebras. This is also a speciality of the discrete setting where singleton sets do not have measure zero.

Definition 3.4 *For $t \in \Lambda$ I set*

$$\begin{aligned} \mathcal{F}_t &= \sigma - alg[\{\omega(s) | s < t\}] \\ &= \sigma - alg[\{\{\omega : \omega(s) = -1\}, \{\omega : \omega(s) = +1\} | s < t\}] \end{aligned}$$

and call this the past algebra*.*
(Note that $\omega(t)$ is not contained in the generating set.)
*A random variable X is said to be $\mathcal{F}_t$-*adapted *if*

$$E[X|\mathcal{F}_t] = X\,.$$

This means that the Walsh decomposition of X has the form

$$X = \sum_{A \subset [0,t[} X(A)\chi_A \quad \textit{with} \quad [0,t[= \{s \in \Lambda : s < t\}\,.$$

A stochastic process $(X_t)_{t \geq 0}$ is adapted *if the random variable X_t is $\mathcal{F}_t$-adapted for each $t \in \Lambda$.* ◁

Thus for a $\mathcal{F}_t$-adapted random variable all Walsh coefficients $X(A)$ with $\max A \geq t$ are zero. Also the Itô integral of an adapted process makes sense since the products of the Walsh components $X_t(A)\chi_A$ of X_t and the forward increment $\triangle B_t = \chi_{\{t\}}\sqrt{\triangle t}$ of the Brownian motion are well defined.

Corollary 3.5 *For every process* $(X_t)_{t\in\Lambda}$ *with Walsh decomposition* $X_t = \sum_{A\subset\Lambda} X(A;t)\chi_A$ *one has*

$$E[X_t|\mathcal{F}_t] = \sum_{A\subset[0,t[} X(A;t)\chi_A = \sum_{\substack{A\subset\Lambda \\ \max A<t}} X(A;t)\chi_A\,.$$

PROOF: Follows from proposition 3.2 and the definition. □

4 Discrete Skorohod integral

Definition 4.1 *Let* $X : \Omega\times\Lambda \to \mathbb{R}$ *be a stochastic process. The* Skorohod integral *of* X *with respect to the Brownian motion* B *is defined by*

$$\int X\delta B = \int X_s\delta B_s = \sum_{s\in\Lambda} X_s \diamond \triangle B_s.$$

◁

As an easy consequence one has

$$\int X_s\delta B_s = \sum_{s\in\Lambda} X_s \diamond \chi_{\{s\}}\sqrt{\triangle t} = \sum_{s\in\Lambda} X_s \diamond W_s\triangle t\,.$$

So the discrete Skorohod integral is the discrete Lebesgue integral of the transformed process by Wick multiplication with pointwise white noise.

The second assertion in the next proposition shows that taking the discrete Wiener-Itô decomposition of a process X_s for each s the Skorohod integral is roughly speaking integration with $\sum_s \cdot\triangle B_s$ over the parameter s.

Proposition 4.2
Let $X_s = \sum_n \sum_{(t_1,\dots,t_n)\in\Lambda^n} X_n(t_1,\dots,t_n;s)\triangle B(t_1)\cdots\triangle B(t_n)$ *be the discrete Wiener-Itô decomposition of the Process* X_s.

(1) *If the stochastic process* X_s *is adapted then the Skorohod integral reduces to the Itô integral.*

(2)

$$\int X_s \delta B_s = \sum_n \sum_{(t_1,\dots,t_{n+1})\in\Lambda^{n+1}} \widehat{X}_{n+1}(t_1,\dots,t_{n+1}) \triangle B(t_1)\cdots\triangle B(t_{n+1})$$

whereby $\widehat{X}_{n+1}(t_1,\dots,t_{n+1})$ *is the symmetrization of the coefficient function* $X_n(t_1,\dots,t_n;s)$ *with respect to the process variable* s.

Proof of 4.2 (1): Take $A<s$ as notation for $\max A < s$. Since X_s is adapted it has the Walsh decomposition $X_s = \sum_{A<s} X(A;s)\chi_A$. Hence A and $\{s\}$ are disjoint and one obtains

$$\begin{aligned}\int X_s \delta B_s &= \sum_s \sum_{A<s} X(A;s)\chi_A \diamond \chi_{\{s\}}\sqrt{\triangle t}\\ &= \sum_s \sum_{A<s} X(A;s)\chi_A \cdot \chi_{\{s\}}\sqrt{\triangle t}\\ &= \sum_s X_s \cdot \triangle B_s = \int X_s dB_s.\end{aligned}$$

□

Proof of 4.2 (2):

$$\begin{aligned}\int X\delta B &= \sum_s X_s \diamond \chi_{\{s\}}\sqrt{\triangle t}\\ &= \sum_s \left(\sum_n \sum_{A\in\mathcal{P}_n} X(A;s)\chi_A\right) \diamond \chi_{\{s\}}\sqrt{\triangle t}\end{aligned}$$

$$= \sum_s \left(\sum_n \sum_{(t_1,\dots,t_n)\in\lambda^n} X_n(t_1,\dots,t_n;s)\chi_{\{t_1,\dots,t_n\}}\triangle t^{\frac{n}{2}}\right) \diamond \chi_{\{s\}}\sqrt{\triangle t}$$

with $X_n(\cdot;s)$ the symmetric functions in the Wiener-Itô decomposition of X_s. Now I rename the parameter $s = t_{n+1}$ and introduce the symmetric functions $\widehat{X}_{n+1}$ of $n+1$ arguments by

$$\begin{aligned}\widehat{X}_{n+1}(t_1,\dots,t_{n+1}) &= 0 \text{ if } t_i = t_j \text{ for some } i\neq j \text{ and}\\ \widehat{X}_{n+1}(t_1,\dots,t_{n+1}) &=\\ \frac{1}{n+1}\Big(\sum_{k=1}^{n+1} X_n(t_1,\dots,t_{k-1},t_{n+1},t_{k+1},\dots,t_n;t_k)\Big) &\text{ otherwise.}\end{aligned}$$

Then one obtains, changing the sum over s inside,

$$\int X\delta B = \sum_n \sum_{(t_1,\dots,t_{n+1})\in\Lambda^{n+1}} \widehat{X}_{n+1}(t_1,\dots,t_{n+1})\chi_{\{t_1,\dots,t_{n+1}\}}\triangle t^{\frac{n+1}{2}}$$

$$= \sum_n \sum_{(t_1,\dots,t_{n+1})\in\Lambda^{n+1}} \widehat{X}_{n+1}(t_1,\dots,t_{n+1})\triangle B(t_1)\cdots\triangle B(t_{n+1})\,.$$

□

So one sees that the discrete Skorohod integral recovers formally the properties of the continuous one.

5 Discrete Malliavin derivative

In this section I will develop a discrete version of the Malliavin derivative. My definition is different from that in [9] but has the right behaviour for the Wiener-Itô decomposition of a random variable.

For the definition of the Malliavin derivative I need the following notation:

Notation 5.1 For $s \in \Lambda$ and $\omega \in \Omega$ I define ω_s^+ and ω_s^- by

$$\omega_s^{\pm}(t) = \begin{cases} \omega(t) & \text{for } t \neq s \\ \pm 1 & \text{for } t = s \end{cases}.$$

◁

Definition 5.2 *For every random variable $X \in L^2(\Omega, P)$ I define the* Malliavin derivative $(D_tX)_{t\geq 0}$ *by the family $(D_t)_{t\geq 0}$ of operators on $L^2(\Omega, P)$:*

$$D_tX(\omega) = \frac{X(\omega_t^+) - X(\omega_t^-)}{2\sqrt{\triangle t}}\,.$$

◁

This family of operators can be seen as an operator

$$D : L^2(\Omega, P) \to L^2(\Omega \times \Lambda, P \times \mu).$$

Remark 5.3 Holden et. al. introduce in [9] an integrated version $\mathbb{D}_t = \sum_{s<t} D_s \triangle t$ of my Malliavin derivative and call this the Malliavin derivative. I would name $\mathbb{D}_t$ the *Malliavin process* and thus one sees that the Malliavin derivative in my sense is the derivative of the Malliavin process: $D_t X(\omega) = \frac{\triangle \mathbb{D}_t X(\omega)}{\triangle t}$. Furthermore Holden et. al. define a discrete Cameron-Martin space and a discrete Malliavin divergence and show that the Malliavin process and the Malliavin divergence are adjoint operators from discrete Wiener space onto discrete Cameron-Martin space.

Proposition 5.4
Let be $X = \sum_n \sum_{(t_1,\ldots,t_n)\in\Lambda^n} X_n(t_1,\ldots,t_n)\triangle B(t_1)\cdots\triangle B(t_n)$. *Then*

$$D_t X = \sum_n \sum_{(t_1,\ldots,t_{n-1})\in\Lambda^{n-1}} n X_n(t_1,\ldots,t_{n-1};t)\triangle B(t_1)\cdots\triangle B(t_{n-1})\,.$$

I.e. the Malliavin derivative acts on the discrete Wiener-Itô decomposition as multiplication by the level number n *and then just leaving aside the integration over* $\Delta B(t_n)$.

PROOF:

$$\begin{aligned}
D_t X(\omega) &= D_t\left(\sum_n \sum_{(t_1<\ldots<t_n)\in\Lambda^n} n! X_n(t_1,\ldots,t_n)\triangle t^{\frac{n}{2}}\chi_{\{t_1,\ldots,t_n\}}(\omega)\right)\\
&= \sum_n \sum_{(t_1<\ldots<t_n)\in\Lambda^n} n! X_n(t_1,\ldots,t_n)\triangle t^{\frac{n}{2}}\cdot\\
&\qquad\qquad \cdot\frac{\chi_{\{t_1,\ldots,t_n\}}(\omega_t^+) - \chi_{\{t_1,\ldots,t_n\}}(\omega_t^-)}{2\sqrt{\triangle t}}\\
&= \sum_n \sum_{(t_1<\ldots<t_n)\in\Lambda^n} n! X_n(t_1,\ldots,t_n)\triangle t^{\frac{n-1}{2}}\cdot\\
&\qquad\qquad \cdot\frac{1}{2}\left(\prod_{s\in\{t_1,\ldots,t_n\}}\omega_t^+(s) - \prod_{s\in\{t_1,\ldots,t_n\}}\omega_t^-(s)\right)\\
&= \sum_n \sum_{\substack{(t_1<\ldots<t_n)\in\Lambda^n\\ t\in\{t_1,\ldots,t_n\}}} n! X_n(t_1,\ldots,t_n)\triangle t^{\frac{n-1}{2}}\chi_{\{t_1,\ldots,t_n\}\setminus\{t\}}(\omega)
\end{aligned}$$

$$= \sum_n \sum_{(t_1<\ldots<t_{n-1})\in\Lambda^{n-1}} n! X_n(t_1,\ldots,t_{n-1},t) \triangle t^{\frac{n-1}{2}} \chi_{\{t_1,\ldots,t_{n-1}\}}(\omega)$$

$$= \sum_n \sum_{(t_1,\ldots,t_{n-1})\in\Lambda^{n-1}} n X_n(t_1,\ldots,t_{n-1};t) \triangle B(t_1)\cdots\triangle B(t_{n-1})\,.$$

□

So the discrete Malliavin derivative acts on the discrete Wiener-Itô decomposition of random variables as expected from the continuous case. One sees also that $D_t\chi_\emptyset = 0$. In quantum mechanics $\chi_\emptyset$ would be the vacuum state and D_t the one particle annihilation operator at time t. Thus there is a deep interconnection between Malliavin calculus and quantum stochastic calculus in the discrete setting.

Proposition 5.5

$$D_t X = X \cdot W_t - X \diamond W_t\,.$$

Proof: If $X = \sum_{A\in\mathcal{P}} X(A)\chi_A$ then

$$\frac{X(\omega_s^+) - X(\omega_s^-)}{2\sqrt{\triangle t}} = \sum_{\substack{A\in\mathcal{P}\\ t\in A}} X(A)\chi_{A\setminus\{t\}}(\omega)(\triangle t)^{-\frac{1}{2}}\,.$$

But since $\chi_A \cdot W_t = \chi_A \diamond W_t$ if A and $\{t\}$ are disjoint one gets

$$\begin{aligned} X\cdot W_t - X\diamond W_t &= \\ \sum_{A\in\mathcal{P}} X(A)\chi_A\cdot\chi_{\{t\}}(\triangle t)^{-\frac{1}{2}} &- \sum_{A\in\mathcal{P}} X(A)\chi_A\diamond\chi_{\{t\}}(\triangle t)^{-\frac{1}{2}} \\ = \sum_{A\in\mathcal{P}} X(A)\chi_{A\triangle\{t\}}(\triangle t)^{-\frac{1}{2}} &- \sum_{\substack{A\in\mathcal{P}\\ t\notin A}} X(A)\chi_{A\cup\{t\}}(\triangle t)^{-\frac{1}{2}} \\ &= \sum_{\substack{A\in\mathcal{P}\\ t\in A}} X(A)\chi_{A\setminus\{t\}}(\triangle t)^{-\frac{1}{2}}\,. \end{aligned}$$

Thus the proposition follows. □

6 Discrete Clark-Ocone formula

Now I am prepared to prove the discrete version of the Clark-Ocone formula. The continuous Clark-Ocone formula for random variables F looks like this:

$$F = E[F] + \int E[D_t F|\mathcal{F}_t] dB_t$$

and can be proved under certain conditions for F. The integral here is an Itô integral. In the discrete version I have not any condition since there are no convergence problems for sums or integrals. Furthermore every operator is a bounded operator whence defined everywhere. The discrete Clark-Ocone formula reads as follows.

Theorem 6.1 *Let be $X \in L^2(\Omega, P)$. Then there holds*

$$X = E[X] + \sum_{t\in\Lambda} E[D_t X|\mathcal{F}_t] \cdot \triangle B_t \,.$$

PROOF: Let $X = \sum_{A\in\mathcal{P}} X(A)\chi_A$ be the Walsh decomposition of X. First remember from proposition 2.8 that $E[X] = X(\emptyset)$. I show that

$$\sum_{t\in\Lambda} E[D_t X|\mathcal{F}_t] \cdot \triangle B_t = X - X(\emptyset) \,.$$

I use the following expression for $D_t X$:

$$\begin{aligned} D_t X(\omega) &= \frac{X(\omega_t^+) - X(\omega_t^-)}{2\sqrt{\triangle t}} \\ &= \sum_{A\in\mathcal{P}} \frac{X(A)}{2\sqrt{\triangle t}} \big(\chi_A(\omega_t^+) - \chi_A(\omega_t^-)\big) \\ &= \sum_{\substack{A\in\mathcal{P} \\ t\in A}} \frac{X(A)}{\sqrt{\triangle t}} \chi_{A\setminus\{t\}}(\omega) \\ &= \sum_{\substack{A\in\mathcal{P} \\ t\notin A}} \frac{X(A\cup\{t\})}{\sqrt{\triangle t}} \chi_A(\omega) \,. \end{aligned}$$

Since the conditional expectation with respect to $\mathcal{F}_t$ cuts the Walsh components χ_A with $A \not\subset [0, t[$ that means, by applying corollary 3.5, that one obtains

$$\begin{aligned} E[D_t X|\mathcal{F}_t] &= \sum_{\substack{A\in\mathcal{P} \\ t\notin A\wedge \max A<t}} \frac{X(A\cup\{t\})}{\sqrt{\triangle t}}\chi_A \\ &= \sum_{\substack{A\in\mathcal{P} \\ \max A<t}} \frac{X(A\cup\{t\})}{\sqrt{\triangle t}}\chi_A\,. \end{aligned}$$

Now I integrate this with $\sum_t \cdot \triangle B_t$ and, since A and $\{t\}$ are disjoint, using $\chi_A \cdot \chi_{\{t\}} = \chi_{A\cup\{t\}}$ there follows

$$\begin{aligned} \sum_{t\in\Lambda} E[D_t X|\mathcal{F}_t]\cdot\triangle B_t &= \sum_{t\in\Lambda}\sum_{\substack{A\in\mathcal{P} \\ \max A<t}} \frac{X(A\cup\{t\})}{\sqrt{\triangle t}}\chi_A\cdot\chi_{\{t\}}\sqrt{\triangle t} \\ &= \sum_{t\in\Lambda}\sum_{\substack{A\in\mathcal{P} \\ \max A<t}} X(A\cup\{t\})\chi_{A\cup\{t\}} \\ &= \sum_{t\in\Lambda}\sum_{\substack{A\in\mathcal{P} \\ \max A=t}} X(A)\chi_A \\ &= \sum_{A\in\mathcal{P}\setminus\emptyset} X(A)\chi_A \\ &= X - X(\emptyset)\,. \end{aligned}$$

Thus the proof of the theorem is done. □

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