the Danish National Research Foundation.

Of particular interest are the questions:
Bardou et al. (1994) ГBardou (1995) and Reichel et al. (1995).
Of particular interest are the questions: ing「and subsequently to a dramatic improvement in efficiency of the cooling; Stochastic considerations have led to a substantially better understand-
 their positions and velocities. Each single atom follows a random trajectoryl mechanically on material objects $\Gamma$ such as atoms meaning that it can change indicated in Figure 1. The basis of the techniques is the fact that light acts of atoms largely concentrated in a very small region around the point $\Gamma$ as up a suitable magnetic field around the point it is possible to hold a cloud directing a number of laser beams towards a chosen point in space and setting field technology $\Gamma$ is a subject area of great current interest in physics. By Cooling and trapping of atoms $\Gamma$ by a combination of laser and magnetic
1 Introduction
 AMS subject classifications: 60K05 60J25 60E07 62E20.
 showed that Lévy flights is the key tool. A review of their analysis, from the point of view
of renewal theory and occupation times for stochastic processes, is given here and some


## O.

MaPhySto ${ }^{1}$, University of Aarhus

## LASER COOLING AND STOCHASTICS



Figure 1. Experimental setup for laser cooling and trapping.

These and related questions are discussed in considerable detail in a forthcoming paper by BardouГBochaudГAspect and Cohen-Tannoudji (1999)Га preliminary version of which has kindly been provided to us by Francois Bardou. (See also Bardou and Castin (1998)). We shall refer to their treatment as the ABBC analysis.

The ABBC analysis led to the heart of the matter but through an approximate analysis $\Gamma$ ab initio. In Sections 4 and 5 we review and extend that work in the light of the theory of renewals and occupation times for stochastic processes. In this we draw on well known results of that theory as expounded $\Gamma$ for instance $\Gamma$ in Bingham $\Gamma$ Goldie and Teugels (1987). Section 2 outlines the physical setting in more detail and Section 3 specifies the resulting stochastic process model for a one-dimensional component of the momentum vector. Some analogous $\Gamma$ but simpler $\Gamma$ models that allow of a fairly detailed analysis are briefly treated in Section 6 Гand the final Section 7 contains concluding remarks.

## 2 Laser cooling

The four most prominent cooling techniques $\Gamma$ listed in the order they arose chronologically $\Gamma$ are Doppler cooling $\Gamma$ Sisyphus cooling $\Gamma$ VSCPT (VelocitySelective Coherent Population Trapping) and Raman ccoling. Doppler cooling and Sisyphus cooling were capable of bringing the temperature down to $1 \mu$ KГapproximatelyГbut lower limits are not achievable by these methods due to a recoil effect. With VSCPT and Raman cooling temperatures of the order 1 nK are reached. These two methods rely heavily on the effects of what in the physics literature has become to be known as 'Lévy flights' $\Gamma$ and which play an important role in many other contexts in physics. In the language of stochastics the effects are those associated to the properties of the stable laws (cf. Sections 4 and 5).

For an atom subjected to VSCPTГthe quantum mechanical description of its behaviour is as a wave function $\psi$ and it is this function that un-
dergoes a random trajectory in Hilbert space $\Gamma$ the stochastic movements being caused by absorption and emission of photons. In this connectionTsee Cohen-Tannoudji Bardou and Aspect (1992) Г Castin and Mølmer (1995) and Mølmer and Castin (1996).

The models to be described and discussed in the following refer mainly to the Raman method. Under that type of experimental setup we have in mind here the momentum of the atom is accurately determined $\Gamma$ in the sense of having a narrow probability distribution (centered on zero) and hence $\Gamma$ due to complementarity $\Gamma$ the position is only vaguely determined. Correspondingly $\Gamma$ the stochastic processes we shall be discussing in the following sections are to be conceived as models for the time behaviour of the momentum rather than the position of the atom. However $\Gamma$ this still means (recall the atomic scales) that with high probability the position will be in a (roughly) spherical region with a diameter of the order of 1 mm or less (the central region in Figure 1).

Laser cooling and trapping makes it possible to measure important physical quantities with unprecedented precision and to study various types of fundamental questions in particle physics $\Gamma$ for instance concerning atom optics $\Gamma$ atom interferometryГatomic clocks $\Gamma$ and high resolution spectroscopy. The 1997 Nobel Price in physics was given for research in this area $\Gamma$ to Steven ChuFClaude Cohen-Tannoudji and William D. Phillips. The three Nobel Prize Lectures Cby Chu (1998) Cohen-Tannoudji (1998) and Phillips (1998) Гare highly readable and informative. An earlierГless technical and very illuminating「discussion was given by Aspect and Dalibar (1994).

For the future $\Gamma$ the techniques hold much promise for the study of 'pure' situations $\Gamma$ such as systems of a small number of atoms in well-defined states exhibiting quantum features.

## 3 Stochastic momentum model

As indicated in Sections 1 and $2 \Gamma$ the basic description of the behaviour of a single atom is in terms of the random 'path' of its wave function $\psi$. Under Raman cooling (and also under VSCPT) the description can for many purposes be reduced to the following type of model for the momentum of the atomГas a function of $t$.

Let $Y_{t}$ be a Markov jump process with state space $\mathbb{R}^{D}$ and transition law $\mu(x, \mathrm{~d} y)$ for jumps from $x$ to $y$. The rate function for the waiting times will be denoted by $\lambda$; in other words $\Gamma$ letting $\tau(y)$ be the generic notation for a waiting time in state $y$ we have that the law of $\tau(y)$ is exponential with mean $\lambda(y)^{-1}$. We write $\Gamma_{A}(t)$ for the occupation time in a set $A$ up till time $t \Gamma i . e$.

$$
\Gamma_{A}(t)=\int_{0}^{t} \mathbf{1}_{A}\left(Y_{s}\right) \mathrm{d} s
$$



Figure 2.Shape of $\lambda(y)$ in model type I with $\gamma=2$.
and $B_{x}(\rho)$ will denote the ball in $\mathbb{R}^{D}$ with radius $\rho$ and center $x$. We shall refer to $B_{0}(r) \Gamma$ for some small $r$ Гas the 'trap' $\Gamma$ this corresponding to the cold states of the atom. Finally $\mathrm{Clet} A_{t}$ be the random variable that is 0 if the atom is in the trap at time $t$ and is 1 otherwise $\Gamma$ and define $q_{t}$ to be the conditional probability density of $Y_{t}$ given that $A_{t}=0 \Gamma$ i.e.

$$
q_{t}(y)=p\left(y \ddagger Y_{t} \mid A_{t}=0\right)
$$

The dimensions $D=1,2$ or 3 are those of physical interest Гand we shall mainly consider the one-dimensional case. The key experimental setting is such thatГup to a scaling「which is unimportant in the present context (see further in Section $4 \Gamma$ Footnote 3) $\Gamma$

$$
\lambda(y)=c|y|^{\gamma} \quad \text { for } y \in B_{0}(1)
$$

for some $c>0$ and some $\gamma>0^{2}$. The parameter $\gamma$ is determined by the experimental setup $\Gamma$ the case $\gamma=2$ being of some special interest ${ }^{3}$. For $y$ outside the ball $B_{0}(1)$ various forms of $\lambda(y)$ are considered. We shall discuss three model types:

Model type $I$ : For some $R>1$ there is a reflecting barrier at the surface of the ball $B_{0}(R)$ Гand $\lambda(y)=c$ for $1<|y|<R$.
Model type II: $\lambda(y)=c$ for all $y$ with $1<|y|$.
Model type III: $\lambda(y)=1$ for $1<|y|<R$ and $\lambda(y)=c(R /|y|)^{\eta}$ for $\eta>0$ and all $y$ with $R \leq|y|$.

[^0]FurthermoreГunder model types II and III $\mu(x, \mathrm{~d} y)$ is of the form

$$
\mu(x, \mathrm{~d} y)=\phi(y-x) \mathrm{d} y
$$

for some function $\phi \Gamma$ which in the one-dimensional case may be taken as

$$
\phi(\cdot)=\delta^{-1} \mathbf{1}_{[-\delta / 2, \delta / 2]}(\cdot)
$$

for some positive $\delta<1$; that is $\Gamma$ for $D=1$ the jump sizes are uniformly distributed between $-\delta / 2$ and $\delta / 2^{4}$. For model type I this form is modified near the reflecting barrierГin a natural fashion. (For model type IConly the behaviour within the ball $B_{0}(R)$ is studied.)

For investigations in physics $\Gamma$ model type I is the most important and we shall mainly consider this. Furthermore $\Gamma$ for simplicity we largely restrict attention to the one-dimensional case $\Gamma$ i.e. $D=1$.

## 4 The ABBC analysis

As already indicated $\Gamma$ in the ABBC approach one considers $\Gamma$ in momentum space $\Gamma$ a small ball $B_{0}(r)$ - the trap - centred at 0 and with radius $r$.

Let $\tau_{1}, \tau_{2}, \ldots$ and $\hat{\tau}_{1}, \hat{\tau}_{2}, \ldots$ denote the successive sojourn times in and out of the trap $\Gamma$ respectively $\Gamma$ the $\tau_{i}$-s constituting an i.i.d. sequence and likewise for the $\hat{\tau}_{i}$-s. It is assumed that with sufficient accuracy one can think of these two sequences as being independent. The degree of accuracy of the implied approximation depends on the size of $\delta$.

It is furthermore argued that provided $r \ll \delta / 2$ one can $\Gamma$ to good approximation Cassume that when the atom jumps into the trap from outside $\Gamma$ the attained momentum $y$ will be uniformly distributed in $B_{0}(r)$. Letting $\lambda(y)$ denote the rate of the exponential waiting time distribution in momentum state $y$ one therefore has that the $\tau_{i}$-s follow the distribution with density

$$
p(x \ddagger \tau)=\left|B_{0}(r)\right|^{-1} \int_{B_{0}(r)} \lambda(y) e^{-\lambda(y) x} d y
$$

where $\left|B_{0}(r)\right|$ is the volume of $B_{0}(r)$ and $\tau$ is a generic random variable having the same distribution as the $\tau_{i}$-s. Let $\alpha=D / \gamma \Gamma$ then

$$
p(x \ddagger \tau) \sim a|x|^{-(1+\alpha)}
$$

for some constant $a$. Thus $\Gamma$ provided $\alpha<2 \Gamma$ the law of the $\tau_{i}$-s belongs to the domain of attraction of a positive $\alpha$-stable distribution with a scaling constant $b$ depending on $a$. We denote the distribution function of this

[^1]positive $\alpha$-stable law ${ }^{5}$ by $S_{\alpha}(x ; b)$. In particular $\Gamma$ if $\gamma=2$ and $D=1 \Gamma$ then $\alpha=1 / 2$ and
$$
p(x \ddagger \tau)=\frac{1}{2} r^{-1} \gamma\left(\frac{3}{2}, r^{2} x\right) x^{-3 / 2}
$$
where $\gamma(a, x)$ is the incomplete gamma function
$$
\gamma(a, x)=\int_{0}^{x} s^{a-1} e^{-s} d s
$$

Hence the $\tau_{i}$-s are in the domain of attraction of the $\frac{1}{2}$-stable law with scaling constant $b=2^{-3} r^{-2}$.

As regards what happens outside the trap $\Gamma$ it is argued that under model type I the $\hat{\tau}_{i}$-s belong to the domain of attraction of the normal law $\Gamma$ while under type II the domain of attraction is again that of a $\frac{1}{2}$-stable law with some scaling constant $\hat{b} \Gamma$ as is indeed plausible in view of well-known probabilistic results. Under model type IIIГ the distribution of the $\hat{\tau}_{i}$-s is argued to belong to the domain of attraction of a $\frac{1}{4}$-stable law when $\eta$ is chosen equal to 2 .

In the calculations below we will frequently refer to the distributions of $\tau_{i}$ and $\hat{\tau}_{i}$ as being $\alpha$-stable and $\hat{\alpha}$-stable respectively $\Gamma$ where $\hat{\alpha} \leq \alpha$ and $\hat{\alpha}, \alpha \in(0,1)$. In model type II we have $\alpha=\hat{\alpha}$ while under model type IIIГ $\hat{\alpha}<\alpha$. Mathematically C the most interesting cases are $\alpha=\hat{\alpha}=1 / 2$ (model type II) and $\alpha=1 / 2 \Gamma \hat{\alpha}=1 / 4$ (model type III).

### 4.1 Occupation times

We consider here the time spent by the atom in the trap between 0 and $t$.
Model type I: Let $\tau_{(t)}$ denote the longest of the periods spent in the trap before time $t$. For $t \rightarrow \infty \Gamma \tau_{(t)}$ is of the order of $t$ (cf. a well-known property of the stable laws) and hence $\Gamma$ in particular $\Gamma \Gamma_{B_{0}(r)}(t) / t \rightarrow 1$.

Model type II: In this case

$$
\Gamma_{B_{0}(r)}(t) / t \rightarrow \frac{b}{b+\hat{b}}
$$

where $b$ and $\hat{b}$ are the scaling constants of $\tau$ and $\hat{\tau} \Gamma$ respectively.
Model type III: In this model $\hat{\alpha}<\alpha$ and hence $\Gamma_{B_{0}(r)}(t) / t \rightarrow 0$.

### 4.2 The 'sprinkling distributions'

To obtain more precise information on the distribution of the momentum $Y_{t}$ at time $t$ the authors derive the 'sprinkling distributions' $\mathcal{S}_{R}$ and $\mathcal{S}_{E}$. In

[^2]the traditional probabilistic terminology and assuming that the atom starts outside the $\operatorname{trap} \Gamma \mathcal{S}_{R}$ and $\mathcal{S}_{E}$ are $\Gamma$ in fact $\Gamma$ the renewal measures corresponding to the sequences $\left\{\hat{\tau}_{1}+\ldots+\tau_{i}+\hat{\tau}_{i}\right\}$ and $\left\{\hat{\tau}_{1}+\tau_{1}+\ldots+\hat{\tau}_{i}+\tau_{i}\right\} \Gamma$ respectively. Denoting the corresponding renewal densities by $u_{R}$ and $u_{E}$ we have
$$
u_{E}(t)=\sum_{i=1}^{\infty} p^{n *}(t \ddagger \hat{\tau}+\tau)
$$
and
$$
u_{R}(t)=p(t \ddagger \hat{\tau})+\int_{0}^{t} u_{E}(t-x) p(x \ddagger \hat{\tau}) \mathrm{d} x
$$

The Laplace transforms of $u_{E}(t)$ and $u_{R}(t)$ are

$$
\int_{0}^{\infty} e^{-\theta t} u_{E}(t) d t=\frac{\bar{L}\{\theta \ddagger \hat{\tau}\} \bar{L}\{\theta \ddagger \tau\}}{1-\bar{L}\{\theta \ddagger \hat{\tau}\} \bar{L}\{\theta \ddagger \tau\}\}}
$$

and

$$
\int_{0}^{\infty} e^{-\theta t} u_{R}(t) d t=\frac{\bar{L}\{\theta \ddagger \hat{\tau}\}}{1-\bar{L}\{\theta \ddagger \hat{\tau}\} \bar{L}\{\theta \ddagger \tau\}}
$$

respectively $\Gamma$ where $\bar{L}\{\theta \ddagger \chi\}$ is the Laplace transform of the random variable $\chi$ at $\theta$.

Now consider model type I. Then $\hat{\tau}+\tau$ belongs to the domain of attraction of a positive $\alpha$-stable law with scale parameter $b$ and $\Gamma$ as the authors show and as follows also from results of Dynkin and Lamperti (see further in Section 5) Гwe then have

$$
u_{E}(t), u_{R}(t) \sim \frac{1}{b \Gamma(\alpha)} t^{-(1-\alpha)}
$$

for $t \rightarrow \infty$. In model type II $\hat{\tau}$ belongs to the domain of attraction of a positive $\alpha$-stable law with scale parameter $\hat{b}$. Thus $\Gamma$

$$
u_{E}(t), u_{R}(t) \sim \frac{1}{(\hat{b}+b) \Gamma(\alpha)} t^{-(1-\alpha)}
$$

when $t \rightarrow \infty$. Under model type IIIT $\hat{\tau}$ belongs to the $\hat{\alpha}$-stable domain $\Gamma$ where $\hat{\alpha}<\alpha$. Hence $\Gamma$

$$
u_{E}(t), u_{R}(t) \sim \frac{1}{\hat{b} \Gamma(\hat{\alpha})} t^{-(1-\hat{\alpha})}
$$

when $t \rightarrow \infty$.

### 4.3 Trapping probabilities

Next the authors discuss the probability of finding an atom in the trap. We give here a similar derivation of this probability: Let $Q(t)=\operatorname{Pr}\left\{A_{t}=0\right\} \Gamma$ i.e. be the probability of finding the atom in the $\operatorname{trap} B_{0}(r)$ at time $t$. We have

$$
\begin{equation*}
Q(t)=G(t)+\int_{0}^{t} p\left(x \ddagger \hat{\tau}_{1}+\tau_{1}\right) Q(t-x) d x \tag{1}
\end{equation*}
$$

where

$$
G(t)=\int_{0}^{t} p(x \ddagger \hat{\tau}) \operatorname{Pr}\{\tau>t-x\} d x
$$

Relation (1) is a renewal equation $\Gamma$ which has the solution

$$
\begin{align*}
Q(t) & =\int_{0}^{t} G(t-x) u_{E}(x) d x  \tag{2}\\
& =\int_{0}^{t} u_{E}(t-x) \int_{0}^{x} p(u \ddagger \hat{\tau}) \operatorname{Pr}\{\tau>x-u\} d u d x
\end{align*}
$$

The Laplace transform of $Q(t)$ takes the form

$$
\int_{0}^{\infty} e^{-\theta t} Q(t) d t=\bar{L}\{\theta \ddagger \hat{\tau}\} \cdot \frac{1-\bar{L}\{\theta \ddagger \tau\})}{\theta} \cdot \frac{\bar{L}\{\theta \ddagger \hat{\tau}\} \bar{L}\{\theta \ddagger \tau\}}{1-\bar{L}\{\theta \ddagger \hat{\tau}\} \bar{L}\{\theta \ddagger \tau\}}
$$

In order to study the asymptotics of $Q(t)$ we need to distinguish between the different model types. First $\Gamma$ consider model type I. Since $\hat{\tau}$ has finite expectation and $\tau$ belongs to the domain of attraction of an $\alpha$-stable law with scale $b \Gamma$

$$
\int_{0}^{\infty} e^{-\theta t} Q(t) d t \sim \theta^{-1}-\mathrm{E}\{\hat{\tau}\}
$$

when $\theta \rightarrow 0$. Hence $\Gamma$ when $t \rightarrow \infty \Gamma Q(t) \sim 1$. In model type II both $\hat{\tau}$ and $\tau$ have distributions in the domain of attraction of an $\alpha$-stable law with $\alpha \in(0,1)$ and scale parameter $\hat{b}$ and $b \Gamma$ respectively. We have $\Gamma$ for small $\theta \Gamma$

$$
\int_{0}^{\infty} e^{-\theta t} Q(t) d t \sim \frac{b}{\hat{b}+b} \theta^{-1}
$$

which implies

$$
Q(t) \sim \frac{b}{\hat{b}+b}
$$

when $t \rightarrow \infty$. Finally $\Gamma$ for model type III $\tau$ and $\hat{\tau}$ have distributions in the domain of an $\alpha$-stable and an $\hat{\alpha}$-stable distribution $\Gamma$ respectively $\Gamma$ where $\hat{\alpha}<\alpha$ and $\hat{\alpha}, \alpha \in(0,1)$. In this case $\Gamma$ the small $\theta$ behaviour will be

$$
\int_{0}^{\infty} e^{-\theta t} Q(t) d t \sim \frac{b}{\hat{b}} \theta^{(\alpha-\hat{\alpha})-1}
$$

which gives the large time asymptotics

$$
Q(t) \sim \stackrel{b}{\hat{b}} \frac{1}{\Gamma(1-(\alpha-\hat{\alpha}))} \cdot t^{-(\alpha-\hat{\alpha})}
$$

Note that formally for $\hat{\alpha}=\alpha$ this expression becomes $Q(t) \sim b / \hat{b} \Gamma$ which differs from the correct results as given for model type II.

### 4.4 Momentum distribution

Finally the authors discuss the distribution of the momentum at time $t$ inside the trap. We have $\Gamma$

$$
\begin{aligned}
p\left(y, 0 \ddagger Y_{t}, A_{t}\right) & =(2 r)^{-1} \sum_{i=1}^{\infty} \int_{0}^{t} p\left(x \ddagger \hat{\tau}_{1}+\tau_{2}+\ldots+\hat{\tau}_{i}\right) \operatorname{Pr}\{\tau(y)>t-x\} d x \\
& =(2 r)^{-1} \int_{0}^{t} \sum_{i=1}^{\infty} p\left(x \ddagger \hat{\tau}_{1}+\tau_{2}+\ldots+\hat{\tau}_{i}\right) \operatorname{Pr}\{\tau(y)>t-x\} d x \\
& =(2 r)^{-1} \int_{0}^{t} \operatorname{Pr}\{\tau(y)>t-x\} u_{R}(x) d x \\
& =(2 r)^{-1} \int_{0}^{t} e^{-(t-x) \lambda(y)} u_{R}(x) d x \\
& =(2 r)^{-1} \int_{0}^{t} e^{-\left(1-\frac{x}{t}\right) t \lambda(y)} u_{R}\left(\frac{x}{t} t\right) \frac{d x}{t} \cdot t \\
& =(2 r)^{-1} \int_{0}^{1} e^{-(1-u) t \lambda(y)} u_{R}(t u) d u \cdot t
\end{aligned}
$$

The asymptotics for $p\left(y, 0 \ddagger Y_{t}, A_{t}\right)$ is easily studied in terms of the asymptotics of $u_{R}$ :

$$
u_{R}(x) \sim c x^{-(1-\tilde{\alpha})}
$$

where $\tilde{\alpha}=\alpha$ in model types I and II and $\tilde{\alpha}=\hat{\alpha}$ in model type III. Furthermore $\Gamma c=(b \Gamma(\alpha))^{-1}$ and $c=((b+\hat{b}) \Gamma(\alpha))^{-1}$ in model types I and IIГ respectively $\Gamma$ and $c=(\hat{b} \Gamma(\hat{\alpha}))^{-1}$ in model type III. Hence $\Gamma$ for $t \rightarrow \infty \Gamma$

$$
\begin{equation*}
p\left(y, 0 \ddagger Y_{t}, A_{t}\right) \sim \frac{c}{2 r} \mathcal{G}_{\tilde{\alpha}}(t \lambda(y)) t^{\tilde{\alpha}} \tag{3}
\end{equation*}
$$

where

$$
\mathcal{G}_{\eta}(x)=\int_{0}^{1} e^{-(1-u) x} u^{\eta-1} d u
$$

Our main interest lies in the large time behaviour of $q_{t}(y)$ : First $\Gamma$ notice that

$$
q_{t}(y)=p\left(y \ddagger Y_{t} \mid A_{t}=0\right)=Q^{-1}(t) p\left(y, 0 \ddagger Y_{t}, A_{t}\right)
$$

Hence $\Gamma$ we can give the asymptotic results for $q_{t}(y)$ under the three different model types appealing to the asymptotics for $Q(t)$ derived in the subsection above: Under model type I

$$
\begin{equation*}
q_{t}(y) \sim(2 r b \Gamma(\alpha))^{-1} \cdot \mathcal{G}_{\alpha}(t \lambda(y)) t^{\alpha}, \quad t \rightarrow \infty \tag{4}
\end{equation*}
$$

For model type IIT

$$
\begin{equation*}
q_{t}(y) \sim \frac{b}{2 r b \Gamma(\alpha)} \mathcal{G}_{\alpha}(t \lambda(y)) t^{\alpha}, \quad t \rightarrow \infty \tag{5}
\end{equation*}
$$

and「finally「for model type IIIT

$$
\begin{equation*}
q_{t}(y) \sim \frac{\Gamma(1-(\alpha-\hat{\alpha}))}{2 r b \Gamma(\alpha)} \mathcal{G}_{\hat{\alpha}}(t \lambda(y)) t^{2 \alpha-\hat{\alpha}}, \quad t \rightarrow \infty \tag{6}
\end{equation*}
$$

It follows that through rescaling by the transformation $u=\beta_{t} y \Gamma$ where $\beta_{t}$ is defined by $t \lambda\left(\beta_{t}^{-1}\right)=1$ Гone obtains a limit law for $u\left(\right.$ conditional on $\left.A_{t}=0\right)$ in all three cases.

## 5 Further analysis

We now return to the two first themes of Section 4 in order to discuss these further in the light of existing probabilistic results on occupation times and renewal theory.

### 5.1 Occupation times

Let us first consider the general momentum model introduced in Section 3. From Ethier and Kurtz (1986)Гp.162Гwe know that $Y_{t}$ is a time-homogeneous Markov process with generator given by

$$
\begin{equation*}
\mathcal{A} f(x)=\lambda(x) \int_{\mathbb{R}^{D}}(f(x)-f(y)) \mu(x, d y) \tag{7}
\end{equation*}
$$

The domain of $\mathcal{A}$ is the space of real-valued measurable functions on $\mathbb{R}^{D}$ which are integrable with respect to the measure $\mu(x, d y)$.

Assume $\lambda(\cdot) \geq 0$ is bounded $\Gamma$ and denote $\lambda:=\sup _{y} \lambda(y)$. Introduce a modification of the transition probabilities $\mu$ in the following manner:

$$
\begin{equation*}
\tilde{\mu}(x, A)=\left(1-\frac{\lambda(x)}{\lambda}\right) \delta_{A}(x)+\frac{\lambda(x)}{\lambda} \mu(x, A) \tag{8}
\end{equation*}
$$

Let $\left\{x_{k}\right\}$ be the Markov chain with transition law $\tilde{\mu}$. According to Ethier and Kurtz (1986) $\Gamma Y_{t}$ has the same finite dimensional probability distributions as the process $X_{t}:=x_{P_{t}} \Gamma$ where $P_{t}$ is a Poisson process with intensity $\lambda$
independent of $\left\{x_{k}\right\}$. The transition probabilites for $X_{t}$ are easily derived to be

$$
\begin{equation*}
\operatorname{Pr}\left\{X_{t+s} \in A \mid X_{t}=x\right\}=e^{\lambda s} \sum_{n=0}^{\infty} \frac{(\lambda s)^{n}}{n!} \operatorname{Pr}\left\{x_{n} \in A \mid x_{0}=x\right\} \tag{9}
\end{equation*}
$$

Our main object of interest is relative occupation times for $Y_{t}$. Denote the occupation time in a Borel set $A$ given that $Y_{0}=x$ by

$$
\begin{equation*}
\Gamma_{A}^{x}(t)=\int_{0}^{t} \mathbf{1}_{A}\left(Y_{s}^{x}\right) d s \tag{10}
\end{equation*}
$$

The relative occupation time in question for laser cooling is

$$
\begin{equation*}
\Gamma_{\mathrm{rel}}^{x}(t):=\Gamma_{B_{0}(r)}^{x} / \Gamma_{B_{0}^{c}(r)}^{x} \tag{11}
\end{equation*}
$$

We consider the occupation time distribution of $Y_{t}$ by exploiting the equivalence between the processes $X_{t}$ and $Y_{t}$ : Let $A \in \mathcal{B}\left(\mathbb{R}^{D}\right)$. For $n \in \mathbb{N}_{0}$ define

$$
\begin{equation*}
N_{A}(n)=\#\left\{x_{i} \in A: 0 \leq i \leq n\right\} \tag{12}
\end{equation*}
$$

i.e. $N_{A}(n)$ is the number of visits to the set $A$ of the Markov chain $\left\{x_{i}\right\}$ up till time $n$. Denote the number of jumps of $X_{t}$ between 0 and $t$ by $N_{t}$ and define

$$
\begin{equation*}
\tilde{N}_{A}(t):=N_{A}\left(N_{t}\right)=\#\left\{x_{i} \in A: 0 \leq i \leq N_{t}\right\} \tag{13}
\end{equation*}
$$

With these objects at hand $\Gamma$ we can start to calculate an expression for the (defective) probability density of the occupation time of $Y_{t}$ in a set $A$. For $s \leq t \Gamma$ let $p^{x_{0}}\left(s \ddagger \Gamma_{A}(t)\right)$ be the (defective) probability density of $\Gamma_{A}(t)$ at $s$ when $Y_{0}=x_{0} \notin A$. If $\tilde{N}_{A}(t)=k$ we know that $X_{s}$ has spent $k$ exponentially distributed time periods in the set $A$ on the time interval $[0, t]$. These exponential waiting times are independent with intensity $\lambda \Gamma$ and the sum of $k$ periods will thus be gamma distributed with parameters $k$ and $\lambda$. Hence $\Gamma$

$$
\begin{equation*}
p^{x_{0}}\left(s \ddagger \Gamma_{A}(t)\right)=p^{x_{0}}\left(0 \ddagger \tilde{N}_{A}(t)\right) \delta_{0}(s)+\sum_{k=1}^{\infty} p^{x_{0}}\left(k \ddagger \tilde{N}_{A}(t)\right) g(s ; k, \lambda) \tag{14}
\end{equation*}
$$

where $g(s ; k, \lambda)=\frac{\lambda^{k}}{\Gamma(k)} s^{k-1} e^{-\lambda s} \Gamma$ is the density of the gamma distribution. A straightforward calculation with conditional probabilities shows that

$$
\begin{aligned}
p^{x_{0}}\left(k \ddagger \tilde{N}_{A}(t)\right) & =\sum_{n=k}^{\infty} p^{x_{0}}\left(k \ddagger N_{A}(n)\right) p\left(n \ddagger N_{t}\right) \\
& =e^{-\lambda t} \sum_{n=k}^{\infty} \frac{(\lambda t)^{n}}{n!} p^{x_{0}}\left(k \ddagger N_{A}(n)\right)
\end{aligned}
$$

since $N_{t}$ is Possion distributed with intensity $\lambda$. Thus

$$
\begin{align*}
p^{x_{0}}\left(s \ddagger \Gamma_{A}(t)\right)= & e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^{n}}{n!} p^{x_{0}}\left(0 \ddagger N_{A}(n)\right) \delta_{0}(s)  \tag{15}\\
& +e^{-\lambda(t+s)} \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \frac{\lambda^{n+k} t^{n} s^{k-1}}{n!(k-1)!} p^{x_{0}}\left(k \ddagger N_{A}(n)\right) \\
= & e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^{n}}{n!} p^{x_{0}}\left(0 \ddagger N_{A}(n)\right) \delta_{0}(s) \\
& +e^{-\lambda(s+t)} \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{\lambda^{n+k} t^{n} s^{k-1}}{n!(k-1)!} p^{x_{0}}\left(k \ddagger N_{A}(n)\right) \\
= & e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^{n}}{n!} p^{x_{0}}\left(0 \ddagger N_{A}(n)\right) \delta_{0}(s) \\
& +e^{-\lambda(s+t)} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n}}{n!} \sum_{k=1}^{n} \frac{\lambda^{k} s^{k-1}}{(k-1)!} p^{x_{0}}\left(k \mp N_{A}(n)\right)
\end{align*}
$$

Hence $\Gamma$ we see that the problem of calculating the occupation time of $Y_{t}$ is reduced to finding the occupation time in $A$ for the chain $\left\{x_{k}\right\}$.

We now consider the asymptotics for the occupation time in the framework provided by Takacs (1959): Assume we have a stochastic process which enters states $A$ and $B$ alternately. The states $A$ and $B$ are disjoint subsets of the state space of the process $\Gamma$ and their union constitutes the whole state space. The sequences of the successive sojourn times spent in the two states are assumed to be independent positive random variables. Under some asymptotic assumptions for the sums of the sojourn times in the two states $\Gamma$ Takacs (1959) provides explicit asymptotic results for the total sojourn time in state $B$ (or $A$ ) during the time interval $(0, t)$. His results are directly applicable to the laser cooling framework. We consider this in further detail:

Let state $B=B_{0}(r)$ where $r \ll 1$ and $\Gamma$ as in the Section above $\Gamma\left\{\tau_{i}\right\}$ denotes the sequence of sojourn times in $B \Gamma$ while the sojourn times in state $A=B^{c}$ are denoted $\left\{\hat{\tau}_{i}\right\}$. As we saw in Section $4 \Gamma$ the $\tau_{i}$-s will belong to the domain of attraction of a positive stable distribution of index $\alpha$ and scale parameter $b$. I.e. $\Gamma$

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \operatorname{Pr}\left\{\frac{\sum_{i=1}^{n} \tau_{i}}{n^{1 / \alpha}} \leq x\right\}=S_{\alpha}(x ; b) \tag{16}
\end{equation*}
$$

Concerning the shape of the distribution of the $\hat{\tau}_{i}$-s $\Gamma$ this will depend on the choice of model. In model type I $\Gamma$ the $\hat{\tau}_{i}$-s will belong to the domain of attraction of the normal distribution:

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \operatorname{Pr}\left\{\frac{\sum_{i=1}^{n} \hat{\tau}_{i}-\mathrm{E}\{\hat{\tau}\} n}{\sigma n^{1 / 2}} \leq x\right\}=\Phi(x) \tag{17}
\end{equation*}
$$

where $\sigma^{2}$ is the variance of the generic variable $\hat{\tau}$ with the same distribution as the $\hat{\tau}_{i}$-s. $\Phi(x)$ is the standard normal distribution. In model type IIГthe $\hat{\tau}_{i}$-s belong to the domain of attraction of a stable law of index $\alpha$ with scale parameter $\hat{b}$ :

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \operatorname{Pr}\left\{\frac{\sum_{i=1}^{n} \hat{\tau}_{i}}{n^{1 / \alpha}} \leq x\right\}=S_{\alpha}(x ; \hat{b}) \tag{18}
\end{equation*}
$$

Finally「for model type IIIГ the sojourn times in the 'hot' state are in the domain of attraction of a stable law $S_{\hat{\alpha}}(x ; \hat{b})$

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \operatorname{Pr}\left\{\frac{\sum_{i=1}^{n} \hat{\tau}_{i}}{n^{1 / \hat{\alpha}}} \leq x\right\}=S_{\hat{\alpha}}(x ; \hat{b}) \tag{19}
\end{equation*}
$$

The conditions (16) and (17-19) are exactly what is needed in order to state the following result by Takacs (1959):

Theorem 5.1 The asymptotics of $\Gamma_{B_{0}(r)}(t)$ is given by

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \operatorname{Pr}\left\{\frac{\Gamma_{B_{0}(r)}(t)-M_{j} t}{\tilde{M}_{j} t^{m_{j}}} \leq s\right\}=Q_{j}(s), \quad j=1,2,3 \tag{20}
\end{equation*}
$$

where,
Model type $I$. $m_{1}=\alpha, M_{1}=1$ and $\tilde{M}_{1}=E\{\hat{\tau}\} . Q_{1}(s)$ is the distribution of $-\chi^{-1 / 2}$ where $\chi$ is distributed as $S_{\alpha}(s ; b)$.

Model type II. $m_{2}=1, M_{2}=0$ and $\tilde{M}_{2}=1 . Q_{2}(s)$ is the distribution of $\chi /(\zeta+\chi)$ where $\zeta$ is distributed as $S_{\alpha}(s ; \hat{b})$ and $\chi$ as $S_{\alpha}(s ; b)$.

Model type III. $m_{3}=\hat{\alpha} / \alpha, M_{3}=0$ and $\tilde{M}_{3}=1 . Q_{3}(s)$ is the distribution of $\chi \zeta^{-1 / 2}$ where $\zeta$ is distributed as $S_{\hat{\alpha}}(s ; \hat{b})$ and $\chi$ as $S_{\alpha}(s ; b)$.

### 5.2 The 'sprinkling' distribution

Again $\Gamma$ let $\left\{\hat{\tau}_{i}\right\}_{1}^{\infty}$ and $\left\{\tau_{i}\right\}_{1}^{\infty}$ be independent random variables denoting the sojourn times in the 'hot' and 'cold' states respectively. Let $F(t)$ and $\hat{F}(t)$ be the distribution functions of $\tau_{i}$ and $\hat{\tau}_{i}$ respectively $\Gamma$ where we assume the tail behaviour

$$
\begin{equation*}
1-F(t) \sim \frac{b \ell(t)}{\Gamma(1-\alpha)} \cdot t^{-\alpha}, \quad t \rightarrow \infty \tag{21}
\end{equation*}
$$

and

$$
\begin{equation*}
1-\hat{F}(t) \sim \frac{\hat{b} \ell(t)}{\Gamma(1-\alpha)} \cdot t^{-\alpha}, \quad t \rightarrow \infty \tag{22}
\end{equation*}
$$

for positive constants $b \Gamma \hat{b}$ and $\alpha$. Note that in the case of model type II we have $\alpha=1 / 2 \Gamma \ell(t)=1 \Gamma$ the constants $b$ and $\hat{b}$ being the scale parameters of
the stable distribution. For convenience $\Gamma$ we denote the distribution function of $\sigma:=\hat{\tau}+\tau$ by $F_{\sigma}$. Further $\Gamma \hat{\tau}$ and $\tau$ are generic random variables distributed as $\hat{\tau}_{i}$ and $\tau_{i}$. Introduce the two renewal processes

$$
\begin{align*}
& \hat{M}_{t}=\max \left\{k \mid \hat{\tau}_{1}+\tau_{1}+\hat{\tau}_{2}+\ldots+\hat{\tau}_{k} \leq t\right\}  \tag{23}\\
& M_{t}=\max \left\{k \mid \hat{\tau}_{1}+\tau_{1}+\hat{\tau}_{2}+\ldots+\hat{\tau}_{k}+\tau_{k} \leq t\right\} \tag{24}
\end{align*}
$$

By convention $\Gamma$ we let $\hat{M}_{t}=0$ and $M_{t}=0$ if the sets of $k$ 's to maximize are empty. Define the process

$$
\begin{equation*}
S_{t}=\sum_{i=1}^{\hat{M}_{t}} \hat{\tau}_{i}+\sum_{i=1}^{M_{t}} \tau_{i} \tag{25}
\end{equation*}
$$

The processes $\hat{M}_{t}$ and $M_{t}$ decide the state of the cooling process. To see this $\Gamma$ introduce the times $R_{1}=\hat{\tau}_{1} \Gamma E_{1}=\hat{\tau}_{1}+\tau_{1} \Gamma R_{2}=\hat{\tau}_{1}+\tau_{1}+\hat{\tau}_{2} \Gamma E_{2}=$ $\hat{\tau}_{1}+\tau_{1}+\hat{\tau}_{2}+\tau_{2} \Gamma \ldots$ The $E_{i}$-s denote the exit timesГi.e. the times when the process exits the cooling state. On the other hand $\Gamma$ the $R_{i}$-s are the times the atom returns to the cooling state. It is easy to see that if $\hat{M}_{t}=n$ and $M_{t}=n-1 \Gamma$ then $t \in\left[R_{n}, E_{n}\right) \Gamma$ while if $\hat{M}_{t}=n$ and $M_{t}=n \Gamma t \in\left[E_{n}, R_{n+1}\right)$. Thus $\Gamma M_{t}$ is either equal to or one less than $\hat{M}_{t}$. In the former case the process $S_{t}$ is in the hot state (i.e. the waiting time to next change is distributed as $\left.\hat{\tau}_{n}\right) \Gamma$ while in the latter $S_{t}$ is in the cooled state (i.e. waiting time to next change is distributed according to $\tau_{n+1}$ ). In the previously introduced notation $\Gamma A_{t}=0$ if and only if $\hat{M}_{t}=M_{t}+1$ while $A_{t}=1$ if and only if $\hat{M}_{t}=M_{t}$.

We consider the asymptotic behaviour of the residual time $R_{t}:=t-$ $S_{t}$ when we are in the 'cool' state $\Gamma$ i.e. when $A_{t}=0$. Motivated by the Dynkin-Lamperti theorem (see Bingham et al. (1987) $\Gamma$ p.361) ) it is natural to consider $R_{t} / t$ and show that this has a limiting distribution. We adopt the argument in Bingham et al. (1987) $\Gamma$ p. $361 \Gamma$ to our case of two independent sequences of waiting times: Let $u \leq v$ and $u, v \in[0,1]$. We have that $u t \leq R_{t} \leq v t$ and $A_{t}=0$ if and only if for some $n \in \mathbb{N}_{0}$ and $y \in[1-v, 1-u] \Gamma$

$$
\sum_{i=1}^{n} \sigma_{i}+\hat{\tau}_{n+1}=t y
$$

and $\tau_{n+1} \geq t(1-y)$. Summing over $n$ and integrating over $y$ we get

$$
\begin{equation*}
\operatorname{Pr}\left\{u \leq R_{t} / t<v, A_{t}=0\right\}=\int_{1-v}^{1-u}(1-F(t(1-y)))\left(\hat{F} * U_{\sigma}\right)(t d y) \tag{26}
\end{equation*}
$$

where $U_{\sigma}(t)$ is the renewal measure associated to $\sigma$. Observe that $\hat{F} * U_{\sigma}=$ $U_{R}$ Гthe 'sprinkling' distribution with density $u_{R}(t)$. The right hand side of
(26) can be written

$$
\begin{aligned}
\int_{1-v}^{1-u}(1-F(t(1-y))) U_{R}(t d y)= & \int_{1-v}^{1-u} \frac{1-F(t(1-y))}{1-F(t)}(1-F(t)) \\
& \times U_{\sigma}(t) \frac{U_{R}(t d y)}{U_{\sigma}(t)}
\end{aligned}
$$

From Feller (1971) p . $271 \Gamma$ we have

$$
1-F * \hat{F}(t) \sim \frac{(b+\hat{b}) \ell(t)}{\Gamma(1-\alpha)} \cdot t^{-\alpha}
$$

when $t \rightarrow \infty$ Гand by Tauberian theory this yields

$$
U_{R}(t) \sim \frac{\Gamma(1+\alpha)}{(b+\hat{b}) \ell(t)} \cdot t^{\alpha}
$$

when $t \rightarrow \infty$. HenceГfor $t \rightarrow \infty \Gamma$

$$
(1-F(t)) U_{\sigma}(t) \sim \frac{b}{b+\hat{b}} \frac{\sin \pi \alpha}{\pi \alpha}
$$

and

$$
\frac{U_{R}(t y)}{U_{\sigma}(t)} \sim \frac{\ell(t)}{\ell(t y)} y^{\alpha} \sim y^{\alpha}
$$

implying

$$
\frac{U_{R}(t d y)}{U_{\sigma}(t)} \sim \alpha y^{\alpha-1}
$$

Finally「

$$
\frac{1-F(t(1-y))}{1-F(t)} \sim \frac{1}{(1-y)^{\alpha}}
$$

In conclusion $\Gamma$ we get

$$
\begin{aligned}
\int_{1-v}^{1-u}(1-F(t(1-y))) U_{R}(t d y) & \rightarrow \int_{1-v}^{1-u} \frac{1}{(1-y)^{\alpha}} \frac{b}{b+\hat{b}} \frac{\sin \pi \alpha}{\pi \alpha} \cdot \alpha y^{\alpha-1} d y \\
& =\frac{b}{b+\hat{b}} \frac{\sin \pi \alpha}{\pi} \int_{u}^{v} y^{-\alpha}(1-y)^{-(1-\alpha)} d y
\end{aligned}
$$

Similar calculations can be worked through for the case when $A_{t}=1$. Hence $\Gamma$ we have the following version of the Dynkin-Lamperti Theorem in the case of two independent sequences $\left\{\tau_{i}\right\}$ and $\left\{\hat{\tau}_{i}\right\}$ :

Theorem 5.2 Assume $F$ and $\hat{F}$ have tail behaviour as in (21) and (22). Then the normalized residual time $R_{t} / t:=1-S_{t} / t$ when we are in the 'cool' state has a limiting distribution

$$
\lim _{t \rightarrow \infty} \operatorname{Pr}\left\{R_{t} / t \in(u, v), A_{t}=0\right\}=\frac{b}{b+\hat{b}} \int_{u}^{v} g_{\alpha}(y) d y
$$

and, when we are in the 'hot' state,

$$
\lim _{t \rightarrow \infty} \operatorname{Pr}\left\{R_{t} / t \in(u, v), A_{t}=1\right\}=\frac{\hat{b}}{b+\hat{b}} \int_{u}^{v} g_{\alpha}(y) d y
$$

where

$$
g_{\alpha}(x)=\frac{\sin \pi \alpha}{\pi} y^{-\alpha}(1-y)^{-(1-\alpha)}
$$

## 6 Some analogous models

The approach considered in Sections 4 and 5 is based on certain approximations in relation to the basic model for the momentum $\Gamma$ specified in Section 3. While these approximations appear highly plausibleГany precise assessment of their accuracy is not available at present.

In this section we briefly discuss two model types that can be considered as alternative approximations to the momentum model and that allow fairly detailed analysis. The first is purely discrete and the other is of the diffusion type.

### 6.1 Discrete circular models

Let $x(t)$ be a semi-Markov process with a finite state space $\mathcal{S}$ consisting of $m+1$ points that we may view as positioned equidistantly around a circle. We talk of $x(t)$ as the position of the atom at time $t \Gamma$ and one of the points in $\mathcal{S}$ Гdenoted $0 \Gamma$ will be considered as the 'trap'. We index the other points in $\mathcal{S}$ as $i= \pm 1, \ldots, \pm k$ if $m=2 k$ and as $i= \pm 1, \ldots, \pm k, k+1$ if $m=2 k+1$. Let $q_{i}$ denote the density of the waiting time distribution at site $i$ and suppose that $q_{-i}=q_{i} \Gamma i=1, \ldots, k$. Furthermore $\Gamma$ we let $\sigma_{0}$ be the recurrence time to $i=0 \Gamma$ i.e. the time it takes for the atom $\Gamma$ having just left $0 \Gamma$ to return to the $\operatorname{trap} \Gamma$ and $\xi_{0}$ will denote the mean value of $\sigma_{0}$.

For brevity we shall consider here only the case $m=2$ and we write $q$ for $q_{1}$. More general settings and more detailed analyses will be discussed in a forthcoming paper. Also $\Gamma$ we assume that the transitions between states follow the symmetric random walk pattern i.e. transition can take place only to one of the two neighbouring sites on the circle $\Gamma$ with equal probaility $\frac{1}{2}$. Finally $\Gamma$ suppose the process starts at site $i=1$ and we let $\Gamma_{0}(t)=$ $\int_{0}^{t} \mathbf{1}_{\{0\}}(x(s)) d s$. The distribution of $\Gamma_{0}(t)$ then has an atom of size $\operatorname{Pr}\left\{\sigma_{0}>\right.$ $t\}$ at 0 while at $u>0$ the probability density of $\Gamma_{0}(t)$ is

$$
\begin{align*}
p\left(u \ddagger \Gamma_{0}(t)\right)= & \sum_{\nu=1}^{\infty}\left(\operatorname{Pr}\left\{\sigma_{0}>\cdot\right\} * p^{* \nu}\left(\cdot \ddagger \sigma_{0}\right)\right)(t-u) q_{0}^{* \nu}(u) \\
& +\bar{Q}_{0}(u) \sum_{\nu=1}^{\infty} p^{* \nu}\left(t-u \ddagger \sigma_{0}\right) q_{0}^{*(\nu-1)}(u) \tag{27}
\end{align*}
$$

where

$$
\bar{Q}_{0}(u)=\int_{u}^{\infty} q_{0}(v) \mathrm{d} v
$$

We can mimic the ABBC treatment of model type I with $c_{0}=0$ by letting $q_{0}$ and $q$ be the densities of a stable law of index $\frac{1}{2}$ and a negative exponential law with parameter $\lambda \Gamma$ while the case where $c_{0}>0$ may be mimicked by instead letting $q_{0}$ be the density of the negative exponential distribution with a parameter $\lambda_{0}<\lambda$. In the latter case $\Gamma$

$$
q_{0}^{* \nu}(u)=\Gamma(\nu)^{-1} \lambda_{0}^{\nu} u^{\nu-1} e^{-\lambda_{0} u}
$$

and

$$
\begin{equation*}
p^{* \nu}\left(t \ddagger \sigma_{0}\right)=\lambda e^{-\lambda t} \cdot \frac{1}{2} \sum_{x=0}^{\infty}\binom{x+\nu-1}{x} \frac{1}{x!}(\lambda t / 2)^{x} \tag{28}
\end{equation*}
$$

It follows that (27) may be rewritten as

$$
\begin{equation*}
p\left(u \ddagger \Gamma_{0}(t)\right)=R(t-u, u)+R_{0}(t-u, u) \tag{29}
\end{equation*}
$$

where $R_{0}(t-u, u)$ tends to 0 at an exponential rate as $t \rightarrow \infty$ and

$$
\begin{aligned}
R(t-u, u)= & \xi_{0} \sum_{\nu=1}^{\infty} p^{* \nu}\left(t-u \ddagger \sigma_{0}\right) q_{0}^{* \nu}(u) \\
& +\bar{Q}_{0}(u) \sum_{\nu=1}^{\infty} p^{* \nu}\left(t-u \ddagger \sigma_{0}\right) q_{0}^{*(\nu-1)}(u) \\
= & \left(\xi_{0}+\mu\right) S(t-u, u)
\end{aligned}
$$

with $\mu=\lambda^{-1}$ and

$$
\begin{aligned}
S(t-u, u)= & \sum_{\nu=1}^{\infty} p^{* \nu}\left(t-u \ddagger \sigma_{0}\right) q_{0}^{*(\nu-1)}(u) \\
= & \frac{1}{2} \lambda_{0} \lambda e^{-\left\{\lambda_{0} u+\lambda(t-u)\right\}} \\
& \cdot \sum_{x=0}^{\infty} \frac{(\lambda(t-u) / 2)^{x}}{x!^{2}} \sum_{\nu=0}^{\infty} \frac{(x+\nu)!}{\nu!} \frac{\left(\lambda_{0} u\right)^{\nu}}{\nu!} \\
= & \frac{1}{2} \lambda_{0} \lambda e^{-\left\{\lambda_{0} u+\lambda(t-u)\right\}} \\
& \cdot \sum_{x=0}^{\infty} \frac{(\lambda(t-u) / 2)^{x}}{x!} M\left(x+1,1, \lambda_{0} u\right)
\end{aligned}
$$

The last factor is a special case of Kummerer's function ${ }^{6} M(a, c, z)$ defined for $c \neq 0,-1,-2, \ldots$ by

$$
\begin{equation*}
M(a, c, z)=\Gamma(c) \mathbf{M}(a, c, z) \tag{30}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbf{M}(a, c, z)=\sum_{s=0}^{\infty} \frac{a(a+1) \ldots(a+s-1)}{c(c+1) \ldots(c+s-1)} \frac{z^{s}}{s!} \tag{31}
\end{equation*}
$$

By formulae (9.03) $\Gamma(9.04)$ and Subsection 10.4 in Olver (1974; Chapter 7) we have $\Gamma$ for $z$ real and tending to $\infty \Gamma$

$$
\begin{equation*}
M(a, c, z) \sim z^{a-c} e^{z} / \Gamma(a) \tag{32}
\end{equation*}
$$

provided neither $a$ nor $1+a-c$ is a negative integer or 0 . ConsequentlyPfor $t \rightarrow \infty$ we have

$$
\begin{aligned}
S(t-u, u) & \sim \frac{1}{2} \lambda_{0} \lambda e^{-\lambda(t-u)} \sum_{x=0}^{\infty} \frac{\left(\lambda_{0} \lambda u(t-u) / 2\right)^{x}}{x!^{2}} \\
& =\frac{1}{2} \lambda_{0} \lambda e^{-\lambda(t-u)} I_{0}\left(t \sqrt{ }\left\{2 \lambda_{0} \lambda u(t-u)\right\}\right)
\end{aligned}
$$

where $I_{0}$ is the Bessel function

$$
I_{0}(z)=\sum_{s=0}^{\infty} \frac{\left(z^{2} / 4\right)^{s}}{s!^{2}}
$$

Hence $\Gamma$ since for $z$ real and tending to $\infty$

$$
I_{0}(z) \sim(2 \pi)^{-1 / 2} z^{-1 / 2} e^{z}
$$

we find

$$
S(t-u, u) \sim(2 \pi)^{-1 / 2} \lambda_{0} \lambda e^{-\lambda(t-u)}\left\{4 \lambda_{0} \lambda u(t-u)\right\}^{-1 / 4} t^{-1 / 2} e^{\left\{2 \lambda_{0} \lambda u(t-u)\right\}^{-1 / 2}}
$$

All in all we therefore have

$$
\begin{equation*}
p\left(u \ddagger \Gamma_{0}(t)\right) \sim \frac{\left(\xi_{0}+\mu\right) \lambda_{0} \lambda}{\sqrt{2 \pi}\left(4 \lambda_{0} \lambda\right)^{1 / 4}}(u(t-u))^{-1 / 4} \cdot t^{-1 / 2} \cdot e^{\left(2 \lambda_{0} \lambda u(t-u)\right)^{-1 / 2}-\lambda(t-u)} \tag{33}
\end{equation*}
$$

for $t \rightarrow \infty$.

[^3]
### 6.2 A diffusion model

We introduce a diffusion model for the atomic momentum which is a geometric Brownian motion in a neighbourhood of zero and a reflected Brownian motion elsewhere. The process will be reflected at $R$ and $-R($ for $R>1)$. Consider the diffusion

$$
\begin{equation*}
d X_{t}=\sqrt{\lambda\left(X_{t}\right)} d B_{t}, \quad X_{0}=x \geq 0 \tag{34}
\end{equation*}
$$

where

$$
\begin{equation*}
\lambda(x)=x^{\alpha} \mathbf{1}_{(-1,1)}(x)+\mathbf{1}_{(-R,-1) \cup(1, R)}(x)+\infty \cdot \mathbf{1}_{(-\infty,-R) \cup(R, \infty)}(x) \tag{35}
\end{equation*}
$$

The diffusion process is symmetric around 0 and converges to zero a.s. when $t \rightarrow \infty$. By construction $\Gamma$ the invariant measure of $X_{t}$ is $\lambda^{-1}(x)$. This model for the laser cooling and trapping process does not take jumps into account. Observe that when $x \in(0,1]$ the first passage time for $X_{t}$ to 1 will have heavy tails. Indeed $\Gamma$ since $X_{t}^{x}=x \exp \left(B_{t}-t / 2\right) \Gamma$ the first passage time to 1 will be the same in distribution as the first passage time to zero of a Brownian motion with drift $-1 / 2$ starting at $\ln x$. Hence $\Gamma$ as is well known $\Gamma$ the tail of the distribution will go like $b t^{-3 / 2}$.

We consider the occupation time for $X_{t}$ in $[-1,1]$. It is sufficient to consider only positive values of $x$ since the process is symmetric around zero and its paths will never cross the axis. Let $\Gamma$

$$
\Gamma^{x}(t):=\int_{0}^{t} \mathbf{1}_{[0,1]}\left(X_{s}^{x}\right) d s
$$

The Laplace transform $z(x)=\int_{0}^{\infty} \exp (-\alpha t) E\left[\exp \left(-\beta \Gamma^{x}(t)\right)\right] d t$ is a piecewise $C^{2}$-solution to

$$
\begin{cases}x^{2} z^{\prime \prime}(x)=2(\alpha+\beta) z(x)-1, & x \in(0,1) \\ z^{\prime \prime}(x)=2 \alpha z(x)-1, & x \in(1, R)\end{cases}
$$

with boundary condition

$$
z^{\prime}(R)=0
$$

(cf. Karatzas and Shreve (1991)). The solution is: For $x \in(0,1] \Gamma$

$$
\begin{aligned}
z(x)= & \frac{\beta}{\alpha(\alpha+\beta)} \cdot \frac{2 \sqrt{2 \alpha} \sinh ((R-1) \sqrt{2 \alpha})}{(1+\sqrt{1+8(\alpha+\beta)}) \cosh ((R-1) \sqrt{2 \alpha})} \times \\
& \frac{1}{1+\frac{2 \sqrt{2 \alpha} \sinh ((R-1) \sqrt{2 \alpha})}{(1+\sqrt{1+8(\alpha+\beta)}) \cosh ((R-1) \sqrt{2 \alpha})}} \cdot \exp \left(\frac{1}{2} \ln x \cdot(1+\sqrt{1+8(\alpha+\beta)})\right)+ \\
& (\alpha+\beta)^{-1}
\end{aligned}
$$

and for $x \in(1, R]$.
$z(x)=\frac{-\beta}{\alpha(\alpha+\beta)} \cdot \frac{1}{1+\frac{2 \sqrt{2 \alpha} \sinh ((R-1) \sqrt{2 \alpha})}{(1+\sqrt{1+8(\alpha+\beta)}) \cosh ((R-1) \sqrt{2 \alpha})}} \cdot \frac{\cosh ((R-x) \sqrt{2 \alpha})}{\cosh ((R-1) \sqrt{2 \alpha})}+\frac{1}{\alpha}$
We can invert these transforms with respect to $\alpha$ :
Theorem 6.1 For $x \in(1, R]$ we have,

$$
\begin{align*}
E^{x}\left[e^{-\beta \Gamma(t)}\right]= & 1+\int_{0}^{t}\left(e^{-\beta(t-s)}-1\right) \\
& \times\left\{\sum_{n=0}^{\infty}(-1)^{n} q_{x, R,+}(\cdot) *\left\{h(\cdot ; \beta) * q_{1, R,-}^{\prime}(\cdot)\right\}^{* n}(\cdot)\right\}(s) d s \tag{36}
\end{align*}
$$

and for $x \in(0,1]$,

$$
\begin{align*}
E^{x}\left[e^{-\beta \Gamma(t)}\right]= & e^{-\beta t}+\sqrt{x} \int_{0}^{t}\left(1-e^{-\beta(t-s)}\right)\left\{\sum_{n=0}^{\infty}(-1)^{n} q_{x, R,+}(\cdot) *\right. \\
& \left.\left\{h(\cdot ; \beta) * q_{1, R,-}^{\prime}(\cdot)\right\}^{*(n+1)} * p_{x}(\cdot ; \beta)\right\}(s) d s \tag{37}
\end{align*}
$$

where

$$
\begin{align*}
& \text { (38) } h(t ; \beta)=\int_{0}^{t} e^{-\beta s}\left\{e^{-s / 8} \frac{2}{\sqrt{2 \pi s}}+\frac{1}{4} \int_{0}^{s} e^{-u / 8} \frac{d u}{\sqrt{2 \pi u}}-\frac{1}{2}\right\} \frac{d s}{\sqrt{2 \pi(t-s)}}  \tag{38}\\
& \text { (39) } q_{x, R, \pm}(t)=\sum_{n=0}^{\infty}(-1)^{n}\left\{p_{x<R}(\cdot) \pm p_{x<R}(\cdot) * p_{1<R}(\cdot)\right\}(\cdot) * p_{1<R}^{* 2 n}(\cdot)(t)  \tag{39}\\
& \text { (40) } p_{x}(t ; \beta)=\frac{1}{4} e^{-\beta t} e^{-t / 8} p_{\frac{1}{2} \ln x<0}(t / 4)
\end{align*}
$$

and $p_{x<R}(t)$ is the density for the first passage time in $R$ for a Brownian motion starting at $x<R$.

We note that it is possible to invert these transforms with respect to $\beta$. In a forthcoming paper we will do this and investigate the asymptotic properties of the distributions.

## 7 Concluding remarks

We hope in the future to address some of the following points.
The ultimate aim would be to give a detailed probabilistic treatment of the path properties of the jump process of the wave function $\psi \Gamma$ which is a stochastic process in an infinite dimensional Hilbert space whose precise properties are determined by quantum mechanics (cf. Section 2).

A less ambitious aim is to analyze the basic momentum model (outlined in Section 3) more directly「that is without the type of initial approximation that lie in treating the successive sojourns in and out of the trap as if they were independent. In particular $\Gamma$ it seems of some considerable interest (in dimensions $D=1,2,3$ ) to obtain more accurate information about: (i) the momentum distribution in the trap (ii) the effect of having the size of the jumps (which is of the order of $\delta$ in the notation we have adopted) comparable to the size of the trap (iii) the relation between time behaviour and ensemble behaviour.

Several of these points seem quite challenging $\Gamma$ but we realize that the interest in them may be largely mathematical rather than motivated by essential physical questions.

Acknowledgements. We are greatly indebted to Francois BardouTSvend Erik GraversenГKlaus MølmerГJan Pedersen and Jon Wellner for extensive and highly helpful discussions.

## REFERENCES

АspectГА. and DalibardГJ. (1994). Le refroidissement des atoms par laser. La Recherche 261 30-37.
BardouГF.ГBouchaudГ J.P.ГЕmileГО.Г АspectГ A. and Cohen-TannoudjiГ C. (1994). Subrecoil laser cooling and Lévy flights. Physical Review Letters 72 203-206.
BardouГF. (1995). Ph D Thesis.
BardouГF.ГBouchaudГ J.P.Г AspectГ A. and Cohen-TannoudjiГ C. (1999). Non-Ergodic Cooling: Subrecoil Laser Cooling and Lévy Statistics. In preparation.
BardouГF. and CastinГY. (1998). Le refroidissement laser subrecul: fonctions d'onde stochastiques et vols de Lévy. Images de Physique (Revue de CNRS) (To appear)
BinghamГN.H.ГGoldieГС.M. and TeugelsГJ.L. (1987). Regular variation. Encyclopedia of Mathematics and its ApplicationsГCambridge University PressГCambridge.
CastinГY. and MølmerГK. (1995). Monte Carlo wave-function analysis of 3D optical molasses. Physical Review Letters 74 3772-3775.
ChuTS. (1998). The manipulation of neutral particles. Reviews of Modern Physics 70 685-706.
Cohen-Tannoudji「C. (1998). Manipulating atoms with photons. Reviews of Modern Physics 70 707-719.

Cohen-TannoudjiГC. BardouГF. and AspectГА. (1992). Review on fundamental processes in laser cooling. In M. Ducloy E. Giacobino and G. Camy (Eds.): Proceedings of TENICOLS '91. World ScientificГ Singapore.
EthierГS.N. and KurtzГT.G. (1986). Markov processes: Characterization and ConvergenceГJ. Wiley \& SonsГNew York.
FellerГW. (1971). An Introduction to Probability Theory and its Applications, Vol II. Second Edition. J. Wiley \& SonsГNew York.
GradstheynГI.S. and RyzhikГI.M. (1965). Table of Integrals, Series and Products. Academic Press $\Gamma$ New York.
KaratzasГI. and ShreveГS.E. (1991). Brownian Motion and Stochastic Calculus. Springer-VerlagएNew York.
MølmerГK. and CastinГY. (1996). Monte Carlo wavefunctions in quantum optics. Quantum and Semiclassical Optics 8 49-72.
OlverГF.W.J. (1974). Introduction to Asymptotics and Special Functions. Academic Press $\Gamma$ New York.
Phillips $\Gamma$ W. D. (1998). Laser cooling and trapping of neutral atoms. Reviews of Modern Physics 70 721-741.
ReichelГ J.ГBardouГF.ГBen DahanГМ.ГPeikГЕ.ГRandГS.ГSalomonГC. and Cohen-TannoudjiГC. (1995). Raman cooling of cesium below 3 nK: New approach inspired by Lévy flight statistics. Physical Review Letters 75 4575-4578.
SamorodnitskyГG. and TaqquГM. S. (1994). Stable Non-Gaussian Random Processes. Chapman and HallएNew York.
Takacs $\Gamma$ L. (1959). On a sojourn time problem in the theory of stochastic processes. Transactions of the American Mathematical Society 93 531-540.

MaPhySto<br>Department of Mathematical Sciences<br>University of Aarhus<br>Ny Munkegade<br>DK-8000 Aarhus C<br>DENMARK<br>oebn@imf.au.dk<br>fredb@imf.au.dk


[^0]:    ${ }^{2}$ For VSCPT cooling $\gamma=2$ while under the Raman technique the value of $\gamma$ is adjustable; in particular, experiments with $\gamma=2$ and 4 have been carried out.
    ${ }^{3}$ In some instances, however, a more realistic specification of $\lambda$ inside the ball $B_{0}(1)$ is as

    $$
    \lambda(y)=c|y|^{\gamma}+c_{0}
    $$

    for $c_{0}>0$ but very small. We shall not consider this possibility further here, but take it up in connection with the discrete model formulation in Subsection 6.1

[^1]:    ${ }^{4}$ In the units chosen here, $\delta$ is of the order of $\hbar|k|$ where $\hbar$ is Planck's constant and $k$ is the optical wave-vector.

[^2]:    ${ }^{5}$ In a standard notation (see, e.g., Samorodnitsky and Taqqu (1994)) this law is denoted $S_{\alpha}(b, \beta, \mu)$ with $\beta=1$ and $\mu=0$.

[^3]:    ${ }^{6}$ This function is also referred to as a degenerate hypergeometric function (cf. Gradstheyn and Ryzhik (1965; p. 1058) who use the notation $\Phi(a, c ; z)$ instead of $M(a, c, z)$ ).

