$$
\begin{aligned}
& \text { the Danish National Research Fo } \\
& { }^{3} \text { Supported by NRF } 2039556 .
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 richness of its model; in particular, infinite numbers and objects can be handled easily and rigorously

 quite different underlying techniques and philosophy, and should have distinct applications. The
 the classical space of white noise and in a space obtained from Robinson's nonstandard analysis.

 of a generalized stochastic process (see e.g. $[3,8,16]$ ). to $t$ (understood in a generalized sense). In many respects, $\delta_{a}\left(B_{t}\right)$ serves as the canonical example
 This process is known to be connected to the local time of Brownian motion at $a$. Indeed, the local

 $\delta_{a}$, we get a random variable with infinite variance. In fact, it is not pointwise ( $\omega$ - wise) defined.



[^0]
The covariance of Donsker's Delta Functions ${ }^{1}$
Fred Espen Benth ${ }^{2} \&$ Siu-Ah $\mathrm{Ng}^{3}$

Also it has many powerful tools readily available, whereas a lot of tools in the nonstandard approach need to be produced from scratch.

The nonstandard approach to generalized functionals is a natural development from the nonstandard model used in studying $L^{2}$-Wiener functionals, since such spaces support many functions beyond those representing just $L^{2}$-Wiener functionals. For instance, for certain pairs of nonstandard functions, although they have no $L^{2}$ meaning, they nevertheless have finite nonstandard inner product, and one expects some interpretations in distribution sense. It should be mentioned that even in the early days, Robinson [17] already used nonstandard methods to study Schwartz distributions and so this is just a continuation in an infinite dimensional setting.

This paper is organized as follows: basic classical and nonstandard ingredients needed for this paper are dealt with in $\S 0.1$ and $\S 0.2$; in $\S 1$ we prove in the nonstandard space that the covariance for certain composite generalized functionals is finite; in $\S 2$ we prove that the covariance between certain Donsker's $\delta$-functions is finite, this is done for both the classical and nonstandard space; in $\S 3$, we calculate and give meaning to the correlation between Donsker's $\delta$-functions for Brownian motion at two different time instants, again this is done in both the classical and nonstandard space; finally $\S 4$ is an appendix on nonstandard background needed for $L^{2}$-Wiener functionals, with references included to motivate the use of the nonstandard space for white noise.

It is hoped that both standard and nonstandard practitioners will find something useful in this paper. We take [8] and [9] as standard reference on classical white noise analysis, and [1] for background in nonstandard analysis.
0.1. Classical prerequisites. We first state the necessary notation and terminology from classical white noise analysis: Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let $T$ denote the interval $\left[0, t_{0}\right]$ or $\mathbb{R}_{+}$. Introduce the Hilbert space $H=L^{2}(T, \mathcal{B}, \mu)$ where $\mathcal{B}$ is the Borel $\sigma$-algebra. We have used the notation $\mu$ for the Lebesgue measure on ( $T, \mathcal{B}$ ). Following Nualart [13], $\{W(h), h \in H\}$ is a centered Gaussian family of random variables with variance $|h|_{L^{2}(T)}^{2}$. This family is characterized by the random variables $W(A)=W\left(\mathbf{1}_{A}\right)$ which takes independent values on disjoint subsets of $T$. Note that $W(h)=\int_{T} h d W$ is the Wiener integral. In the sequel we shall use the notation $L^{2}(\Omega)$ for the space $L^{2}(\Omega, \mathcal{G}, P)$ where $\mathcal{G}$ is the $\sigma$-algebra generated by $\{W(A), A \in \mathcal{B}\}$. Elements of $L^{2}(\Omega)$ can be expanded into a series of multiple Wiener integrals (the so-called chaos expansion of the random variable);

Theorem 1. Let $f \in L^{2}(\Omega)$. Then

$$
\begin{equation*}
f=\sum_{n=0}^{\infty} I_{n}\left(f_{n}\right) \tag{1}
\end{equation*}
$$

where $I_{n}$ is the $n$-fold Wiener integral and $f_{n} \in L^{2}\left(T^{n}\right)$ is symmetric. The functions $f_{n}$ are uniquely defined by $f$. Moreover,

$$
\begin{equation*}
\|f\|^{2}=\sum_{n=0}^{\infty} n!\left|f_{n}\right|_{L^{2}\left(T^{n}\right)}^{2}<\infty \tag{2}
\end{equation*}
$$

where $\|\cdot\|$ denotes the usual norm in $L^{2}(\Omega)$.

For a proof of this result, see e.g. [13].
We introduce the Number operator $N$ : If $f \in L^{2}(\Omega)$ has the chaos expansion $f=\sum_{n=0}^{\infty} I_{n}\left(f_{n}\right)$, the application of $N$ on $f$ is defined as

$$
\begin{equation*}
N f=\sum_{n=0}^{\infty} n I_{n}\left(f_{n}\right) \tag{3}
\end{equation*}
$$

The domain of this operator, denoted $\operatorname{Dom}(N)$, is easily seen to be the subspace of $L^{2}(\Omega)$ for which $\sum_{n=0}^{\infty} n!n\left|f_{n}\right|^{2}<\infty$.

Potthoff and Timpel [16] study the following spaces of smooth and generalized random variables: For $\lambda \geq 0$, let $\mathcal{G}_{\lambda}$ be the space of $f \in L^{2}(\Omega)$ where

$$
\begin{equation*}
\|f\|_{\lambda}:=\left\|e^{\lambda N} f\right\| \tag{4}
\end{equation*}
$$

The space of smooth random variables $(\mathcal{G})$ is the projective limit of $\mathcal{G}_{\lambda}$. The space of generalized random variables $(\mathcal{G})^{*}$ is the topological dual to this space. Note that $(\mathcal{G})^{*}$ is the projective limit of $\mathcal{G}_{-\lambda}$, where $\mathcal{G}_{-\lambda}$ is the dual of $\mathcal{G}_{\lambda}$. In fact, this dual space is a Hilbert space with norm $\|\cdot\|_{-\lambda}=$ $\left\|e^{-\lambda N} \cdot\right\|$.
0.2. Nonstandard prerequisites. Next we fix some notation and terminology from nonstandard analysis. More background material needed for $L^{2}$-Wiener functionals can be found in the appendix §4. As in the appendix, we let $N \in{ }^{*} \mathbb{N} \backslash \mathbb{N}, \Delta t=1 / N$ and $\mathbf{T}=\left\{H \Delta t \mid H \in\left[-N^{2}, N^{2}\right] \cap{ }^{*} \mathbb{N}\right\}$, the hyperfinite time line. We define $\bar{\Omega}={ }^{*} \mathbb{R}^{\mathbf{T}}$, with an internal probability measure $\bar{\mu}$ on $\bar{\Omega}$ given by $\mathbf{T}$ independent copies of Gaussian measures, each with distribution $\mathcal{N}(0, \Delta t)$. Elements in $\bar{\Omega}$ are written as $\omega=\left(\omega_{t}\right)_{t \in T}$. So for an internal measurable function $f: \bar{\Omega} \rightarrow{ }^{*} \mathbb{R}$, we have

$$
\begin{equation*}
\mathrm{E}[f]=(2 \pi \Delta t)^{-\frac{N^{2}+1}{2}} \int_{\bar{\Omega}} f(\omega) \exp \left(-\frac{1}{2 \Delta t} \sum_{t \in T} \omega_{t}^{2}\right) d \omega \tag{5}
\end{equation*}
$$

We use $\mathcal{S}\left(\mathbb{R}^{n}\right)$ to denote the Schwartz space of rapidly decreasing functions on $\mathbb{R}^{n}$ and $\mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ to denote its dual, the space of tempered distributions. (See [18].) It can be proved that for every $\phi \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$, there is an internal function $F_{\phi}:{ }^{*} \mathbb{R}^{n} \rightarrow{ }^{*} \mathbb{R}$ (in fact we can even take $F_{\phi} \in{ }^{*} \mathcal{S}\left(\mathbb{R}^{n}\right)$ ) such that

$$
\begin{equation*}
\phi(\xi)=\circ \int_{* \mathbb{R}^{n}} F_{\phi}(x)^{*} \xi(x) d x, \quad \text { for all } \quad \xi \in \mathcal{S}\left(\mathbb{R}^{n}\right) \tag{6}
\end{equation*}
$$

For convenience, we simply identify such $F_{\phi}$ with $\phi$, and omit the ${ }^{*}$, then the above is written as

$$
\begin{equation*}
\phi(\xi)=\circ \int_{* \mathbb{R}^{n}} \phi(x) \xi(x) d x \tag{7}
\end{equation*}
$$

For example, in Theorem 2 in the next section, the ${ }^{*}$ is omitted from $\rho_{i}$ without causing any confusion since it is clear that we are referring to an internal function with domain ${ }^{*} \mathbb{R}$.

We refer to [12] for more details on the use of $\bar{\Omega}$ as a model for white noise. (In [12], only comparison with the Hida space is given, but the general idea is the same.) The only relevant part here is the following: given any generalized functional $\zeta$, there is a function $Z: \bar{\Omega} \rightarrow{ }^{*} \mathbb{R}$ representing $\zeta$ in the sense that for every test functional $\xi$ and any $S L^{2}$-lifting $\Xi$ of $\xi$, the pairing is given by $\zeta(\xi)={ }^{\circ} \mathrm{E}[Z \Xi]$.

## 1. Finiteness of the covariance

We first consider generalized functionals represented by $\phi(X)$ where $\phi \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ and $X$ is a random variable. Note that Donsker's $\delta$-function is a special case of these. (See also [8] Example 3.4 for this kind of generalized functionals.) We prove a simple general result showing that the covariance between $\phi(X)$ and $\theta(X+Y)$ is finite whenever $X, Y$ are independent with their distributions given by rapidly decreasing densities.

Theorem 2. Let $\phi, \theta \in \mathcal{S}^{\prime}(\mathbb{R})$. Let $X, Y: \bar{\Omega} \rightarrow{ }^{*} \mathbb{R}$ be independent random variables, whose distributions are given by densities $\rho_{1}, \rho_{2}$ such that $\rho_{1}(x) \rho_{2}(y-x) \in \mathcal{S}\left(\mathbb{R}^{2}\right)$. (Note that this is satisfied when $\rho_{1}, \rho_{2} \in \mathcal{S}(\mathbb{R})$.) Then both $\mathrm{E}[\phi(X) \theta(X+Y)]$ and $\operatorname{Cov}[\phi(X), \theta(X+Y)]$ are finite.

Proof. By independence and densities, we have

$$
\begin{aligned}
\mathrm{E}[\phi(X) \theta(X+Y)] & =\int_{* \mathbb{R}^{2}} \phi(x) \theta(x+y) \rho_{1}(x) \rho_{2}(y) d x d y \\
& =\int_{* \mathbb{R}^{2}} \phi(x) \theta(y) \rho_{1}(x) \rho_{2}(y-x) d x d y
\end{aligned}
$$

which is finite under the assumptions. Taking $\theta \equiv 1$, we get finiteness of $\mathrm{E}[\phi(X)]$. Taking $\phi \equiv 1$, we get finiteness of $\mathrm{E}[\theta(X+Y)]$. So

$$
\operatorname{Cov}[\phi(X), \theta(X+Y)]=\mathrm{E}[\phi(X) \theta(X+Y)]-\mathrm{E}[\phi(X)] \mathrm{E}[\theta(X+Y)]
$$

is also finite.

The following example shows the limitation on further generalization of the above result.

Example 3. Let $\theta(x)=\exp \left(\frac{x^{2}}{2}\right)(1+|x|)^{2}$. Then for any $\phi \in \mathcal{S}^{\prime}(\mathbb{R})$ and $X, Y \sim \mathcal{N}(0,1), \mathrm{E}[\phi(X) \theta(X+Y)]$ is infinite. To prove this, we simply check that $\int_{*_{\mathbb{R}}} \theta(x+y) \exp \left(-\frac{y^{2}}{2}\right) d y$ is infinite for any $x \not \approx 0$. Of course here $\theta \notin \mathcal{S}^{\prime}(\mathbb{R})$.

Note that in general the conclusion in Theorem 2 relies on the particular representation of the generalized functional. It is possible that $\Phi: \bar{\Omega} \rightarrow{ }^{*} \mathbb{R}$ and $\phi(X)$ both represent the same standard generalized functional but while $\mathrm{E}[\phi(X) \theta(X+Y)]$ is finite, $\mathrm{E}[\Phi \theta(X+Y)]$ is infinite. This seems to be related to choice of the regularization procedure in the classical setting.

## 2. The covariance between Donsker's $\delta$-functions

2.1. In classical white noise space. Consider Donsker's $\delta$-function

$$
\begin{equation*}
\delta_{a}(\phi):=\delta_{a}\left(I_{1}(\phi)\right) \tag{8}
\end{equation*}
$$

for $a \in \mathbb{R}$ constant and $\phi \in L^{2}(\mathbb{R})$. From [8] we know that the chaos expansion of $\delta_{a}(\phi)$ is given by

$$
\begin{equation*}
\delta_{a}(\phi)=\frac{\pi^{1 / 4}}{\sqrt{2 \pi|\phi|^{2}}} e^{-\frac{a^{2}}{4|\phi|^{2}}} \sum_{n=0}^{\infty} \frac{\xi_{n}\left(\frac{a}{\sqrt{2|\phi|^{2}}}\right)}{\sqrt{n!|\phi|^{2}}} I_{n}\left(\phi^{\otimes n}\right) \tag{9}
\end{equation*}
$$

where $\xi_{n}$ is the Hermite function of order $n$ (see e.g. [8]). In [16] it is shown that $\delta_{a}(\phi) \in \mathcal{G}_{-\lambda}$ for all $\lambda>0$. Note that $\delta_{a}\left(B_{t}\right) \notin \mathcal{G}_{0}=L^{2}(\mu)$. However, by the definition of the $\mathcal{G}_{\lambda}$-spaces, we have $\exp (-\lambda N) \delta_{a}(\phi) \in L^{2}(\mu)$ which is the basic property we shall use to discuss covariance for Donsker's $\delta$-function. If we choose $\phi=\mathbf{1}_{[0, t)}$ we obtain Donsker's $\delta$-function for Brownian motion, which we shall denote $\delta_{a}\left(B_{t}\right)$.

For our purposes it will be convenient to use an integral representation of $\delta_{a}(\phi)$ : From [8] we can write

$$
\begin{equation*}
\delta_{a}(\phi)=\frac{1}{2 \pi} \int_{\mathbb{R}} \operatorname{Exp} I_{1}(-i p \phi) \exp \left(i p a-\frac{1}{2} p^{2}|\phi|^{2}\right) d p \tag{10}
\end{equation*}
$$

where $\operatorname{Exp} I_{1}(\psi)$ is the stochastic exponential defined as $\operatorname{Exp} I_{1}(\psi)=\sum_{n=0}^{\infty} \frac{1}{n!} I_{n}\left(\psi^{\otimes n}\right)$. The integral makes sense as a Bochner integral on $\mathbb{R}$ in $\mathcal{G}_{-\lambda}$ for every $\lambda>0$.

In order to define covariance between Donsker's $\delta$-functions, we need to regularize the generalized random variables $\delta_{a}(\phi)$. We shall use the canonical regularization suggested by the definition of $\mathcal{G}_{-\lambda}$ : For each $\lambda>0$, we consider $\exp (-\lambda N) \delta_{a}(\phi)$ which is known to be in $L^{2}(\mu)$. Note that such a regularization is not necessary in a nonstandard framework, as will be seen from the calculations in the next subsection.

We introduce the $\lambda$-covariance for $\delta_{a}(\phi)$ :

Definition 4. For $\lambda>0$ and $\phi, \psi \in L^{2}(\mathbb{R})$, let the $\lambda$-covariance between $\delta_{a}(\phi)$ and $\delta_{b}(\psi)$ be defined as

$$
\begin{equation*}
\operatorname{Cov}_{\lambda}\left[\delta_{a}(\phi), \delta_{b}(\psi)\right]=\operatorname{Cov}\left[e^{-\lambda N} \delta_{a}(\phi), e^{-\lambda N} \delta_{b}(\psi)\right] \tag{11}
\end{equation*}
$$

for $t, s>0$.

The generalized covariance between $\delta_{a}(\phi)$ and $\delta_{b}(\psi)$ can be defined as follows,

Definition 5. For $\lambda>0$ and $\phi, \psi \in L^{2}(\mathbb{R})$. The generalized covariance between $\delta_{a}(\phi)$ and $\delta_{b}(\psi)$ is defined as

$$
\begin{equation*}
\overline{\operatorname{Cov}}\left[\delta_{a}(\phi), \delta_{b}(\psi)\right]:=\lim _{\lambda \downarrow 0} \operatorname{Cov}_{\lambda}\left[\delta_{a}(\phi), \delta_{b}(\psi)\right] \tag{12}
\end{equation*}
$$

whenever this limit exists.
Remark 6. The $\lambda$-covariance is the (classical) covariance between the two ( $L^{2}$ )-random variables $e^{-\lambda N} \delta_{a}(\phi)$ and $e^{-\lambda N} \delta_{b}(\psi)$. Since $\delta_{a}(\phi) \in \mathcal{G}_{-\lambda}$ for every $\lambda>0$, this covariance is finite and thus shows the well-definedness of the $\lambda$-covariance for Donsker's $\delta$-function. It is also straightforward to see that the $\lambda$-covariance is monotonically decreasing for increasing $\lambda$. The generalized covariance, on the other hand, may not exist which is easily seen by putting e. g. $\phi=\psi: \mathrm{E}\left[\left(\exp (-\lambda N) \delta_{a}(\phi)\right)^{2}\right] \rightarrow$ $\infty$ when $\lambda \downarrow 0$. This singularity is not removed by subtracting the square of the expectation of $\exp (-\lambda N) \delta_{a}(\phi)$, since this expectation is finite for $\lambda=0$.

Before we state our main results we introduce the notation

$$
\begin{equation*}
p_{\sigma^{2}}(x)=\frac{1}{\sqrt{2 \pi \sigma^{2}}} \exp \left(-\frac{x^{2}}{2 \sigma^{2}}\right) \tag{13}
\end{equation*}
$$

for the density function for a centered Gaussian random variable with variance $\sigma^{2}$. We have the following theorem:

Theorem 7. Let $\phi, \psi \in L^{2}(\mathbb{R})$ where $(\phi, \psi)=0,|\phi|,|\psi|>0$ and $a, b \in \mathbb{R}$. Then, for arbitrary $\lambda>0$

$$
\begin{equation*}
\operatorname{Cov}_{\lambda}\left[\delta_{a}(\phi), \delta_{b}(\phi+\psi)\right]=\frac{1}{2 \pi} \frac{\exp \left(-\frac{1}{2} f_{\lambda}(a, b)\right)}{\sqrt{|\phi|^{2}|\psi|^{2}+\left(1-e^{-4 \lambda}\right)|\phi|^{4}}}-p_{|\phi|^{2}}(a) p_{|\phi|^{2}+|\psi|^{2}}(b) \tag{14}
\end{equation*}
$$

where

$$
f_{\lambda}(a, b)=\frac{b^{2}}{|\phi|^{2}+|\psi|^{2}}+\frac{|\phi|^{2}+|\psi|^{2}}{|\phi|^{2}|\psi|^{2}+\left(1-e^{-4 \lambda}\right)|\phi|^{4}}\left(a-b e^{-2 \lambda} \frac{|\phi|^{2}}{|\phi|^{2}+|\psi|^{2}}\right)^{2}
$$

Moreover, the generalized covariance is

$$
\begin{equation*}
\overline{\operatorname{Cov}}\left[\delta_{a}(\phi), \delta_{b}(\phi+\psi)\right]=p_{|\phi|^{2}}(a)\left(p_{|\psi|^{2}}(a-b)-p_{|\phi|^{2}+|\psi|^{2}}(b)\right) \tag{15}
\end{equation*}
$$

Proof. Note first that $(\phi, \phi+\psi)=|\phi|^{2}$ and $|\phi+\psi|^{2}=|\phi|^{2}+|\psi|^{2}$. Moreover,

$$
\begin{aligned}
\exp (-\lambda N) \delta_{a}(\phi) & =\frac{1}{2 \pi} \int_{\mathbb{R}} \exp (-\lambda N) \operatorname{Exp} I_{1}(-i p \phi) \exp \left(i p a-\frac{1}{2} p^{2}|\phi|^{2}\right) d p \\
& =\frac{1}{2 \pi} \int_{\mathbb{R}} \operatorname{Exp} I_{1}\left(-i p e^{-\lambda} \phi\right) \exp \left(i p a-\frac{1}{2} p^{2}|\phi|^{2}\right) d p
\end{aligned}
$$

We now calculate $\mathrm{E}\left[\exp (-\lambda N) \delta_{a}(\phi) \exp (-\lambda N) \delta_{b}(\phi+\psi)\right]$ :

$$
\begin{aligned}
& \mathrm{E}\left[\exp (-\lambda N) \delta_{a}(\phi) \exp (-\lambda N) \delta_{b}(\phi+\psi)\right] \\
& \quad=\frac{1}{4 \pi^{2}} \int_{\mathbb{R}^{2}} \mathrm{E}\left[\operatorname{Exp} I_{1}\left(-i p e^{-\lambda} \phi\right) \operatorname{Exp} I_{1}\left(-i q e^{-\lambda}(\phi+\psi)\right)\right] \exp \left(i p a-\frac{1}{2} p^{2}|\phi|^{2}\right) \exp \left(i q b-\frac{1}{2} q^{2}|\phi+\psi|^{2}\right) d p d q \\
& \quad=\frac{1}{4 \pi^{2}} \int_{\mathbb{R}^{2}} \exp \left(-p q e^{-2 \lambda}|\phi|^{2}\right) \exp \left(i p a-\frac{1}{2} p^{2}|\phi|^{2}\right) \exp \left(i q b-\frac{1}{2} q^{2}|\phi+\psi|^{2}\right) d p d q \\
& \quad=\frac{1}{2 \pi} \frac{\exp \left(-\frac{1}{2} f_{\lambda}(a, b)\right)}{\sqrt{|\phi|^{2}|\psi|^{2}+\left(1-e^{-4 \lambda}\right)|\phi|^{4}}}
\end{aligned}
$$

and hence the first part of the theorem is proved since

$$
\mathrm{E}\left[\exp (-\lambda N) \delta_{a}(\phi)\right]=p_{|\phi|^{2}}(a)
$$

By taking the limit when $\lambda \downarrow 0$, we see that the second part follows.
The following corollaries are easily derived,

Corollary 8. Let $a \in \mathbb{R}$ and $\phi, \psi \in L^{2}(\mathbb{R})$ where $(\phi, \psi)=0,|\phi|,|\psi|>0$. Then

$$
\begin{equation*}
\overline{\operatorname{Cov}}\left[\delta_{a}(\phi), \delta_{a}(\phi+\psi)\right]=p_{|\phi|^{2}}(a)\left(\frac{1}{\sqrt{2 \pi|\psi|^{2}}}-p_{|\phi|^{2}+|\psi|^{2}}(a)\right) \tag{16}
\end{equation*}
$$

Corollary 9. For $0<t<s$,

$$
\begin{equation*}
\overline{\operatorname{Cov}}\left[\delta_{a}\left(B_{t}\right), \delta_{a}\left(B_{s}\right)\right]=p_{t}(a)\left(\frac{1}{\sqrt{2 \pi(t-s)}}-p_{s}(a)\right) \tag{17}
\end{equation*}
$$

Proof. Choose $\phi=\mathbf{1}_{[0, t)}$ and $\psi=\mathbf{1}_{[t, s)}$ in (15).
For Donsker's $\delta$-function at zero of Brownian motion, we get the following simple expression for the generalized covariance,

$$
\begin{equation*}
\overline{\operatorname{Cov}}\left[\delta_{0}\left(B_{t}\right), \delta_{0}\left(B_{s}\right)\right]=\frac{1}{2 \pi \sqrt{t}}\left(\frac{1}{\sqrt{t-s}}-\frac{1}{\sqrt{s}}\right) \tag{18}
\end{equation*}
$$

whenever $0<t<s$.
2.2. In the nonstandard space. Let $\epsilon \approx 0$ be a fixed positive infinitesimal. We use $p_{\epsilon}$ (recall (13) ) as a lifting of the Dirac point measure, that is,

$$
\begin{equation*}
\bar{\delta}_{0}(r)=\frac{1}{\sqrt{2 \pi \epsilon}} \exp \left(-\frac{r^{2}}{2 \epsilon}\right) \tag{19}
\end{equation*}
$$

The Dirac point measure at finite $a \in{ }^{*} \mathbb{R}$ is given by $\bar{\delta}_{a}(r)=\bar{\delta}_{0}(r-a)$. Note that for any $\xi \in \mathcal{S}(\mathbb{R})$, one can verify directly that

$$
\begin{equation*}
\xi(a) \approx \int_{* \mathbb{R}} \bar{\delta}_{a}(x) \xi(x) d x \tag{20}
\end{equation*}
$$

(See also [17].)

Lemma 10. Let $X$ and $Y$ be independent centered Gaussian random variables with respective finite variance $\sigma_{1}$ and $\sigma_{2}$, both bounded away from 0 , and let $a, b \in{ }^{*} \mathbb{R}$ be finite, then we have:

$$
\begin{equation*}
\mathrm{E}\left[\bar{\delta}_{a}(X) \bar{\delta}_{b}(X+Y)\right] \approx \frac{1}{2 \pi \sqrt{\sigma_{1} \sigma_{2}}} \exp \left(-\frac{(a-b)^{2} \sigma_{1}+a^{2} \sigma_{2}}{2 \sigma_{1} \sigma_{2}}\right) \tag{21}
\end{equation*}
$$

Proof. We apply Theorem 2 with $\phi=\bar{\delta}_{a}, \quad \theta=\bar{\delta}_{b}, \quad \rho_{1}(x)=\frac{1}{\sqrt{2 \pi \sigma_{1}}} \exp \left(-\frac{1}{2 \sigma_{1}} x^{2}\right) \quad$ and $\rho_{2}(y)=$ $\frac{1}{\sqrt{2 \pi \sigma_{2}}} \exp \left(-\frac{1}{2 \sigma_{2}} y^{2}\right)$. Then

$$
\mathrm{E}\left[\bar{\delta}_{a}(X) \bar{\delta}_{b}(X+Y)\right] \approx \frac{1}{2 \pi \sqrt{\sigma_{1} \sigma_{2}}} \int_{\star_{\mathbb{R}}} \bar{\delta}_{a}(x) \bar{\delta}_{b}(y) \exp \left(-\frac{1}{2 \sigma_{1}} x^{2}\right) \exp \left(-\frac{1}{2 \sigma_{2}}(y-x)^{2}\right) d x d y
$$

and the result follows by noting that $\bar{\delta}_{a}(x) \bar{\delta}_{b}(y)$ represents Dirac point measure at $(a, b)$.

Remark 11. One can compute more accurately the above as

$$
\begin{equation*}
\mathrm{E}\left[\delta_{a}(X) \delta_{b}(X+Y)\right]=\frac{1}{2 \pi \sqrt{\epsilon^{2}+2 \epsilon \sigma_{1}+\epsilon \sigma_{2}+\sigma_{1} \sigma_{2}}} \exp \left(-\frac{(a-b)^{2} \sigma_{1}+a^{2}\left(\epsilon+\sigma_{2}\right)+b^{2} \epsilon}{2\left(\epsilon^{2}+2 \epsilon \sigma_{1}+\epsilon \sigma_{2}+\sigma_{1} \sigma_{2}\right)}\right) \tag{22}
\end{equation*}
$$

Theorem 12. Let $X$ and $Y$ be independent centered Gaussian random variables with respective finite variance $\sigma_{1}$ and $\sigma_{2}$, both bounded away from 0 , and let $a, b \in{ }^{*} \mathbb{R}$ be finite, then we have:

$$
\begin{aligned}
& \operatorname{Cov}\left[\bar{\delta}_{a}(X), \bar{\delta}_{b}(X+Y)\right] \\
\approx & \frac{1}{2 \pi \sqrt{\sigma_{1}}}\left(\frac{1}{\sqrt{\sigma_{2}}} \exp \left(-\frac{(a-b)^{2} \sigma_{1}+a^{2} \sigma_{2}}{2 \sigma_{1} \sigma_{2}}\right)-\frac{1}{\sqrt{\sigma_{1}+\sigma_{2}}} \exp \left(-\frac{a^{2}}{2 \sigma_{1}}-\frac{b^{2}}{2\left(\sigma_{1}+\sigma_{2}\right)}\right)\right)
\end{aligned}
$$

When $a \approx b$, we have:

$$
\begin{equation*}
\operatorname{Cov}\left[\bar{\delta}_{a}(X), \bar{\delta}_{b}(X+Y)\right] \approx \frac{1}{2 \pi \sqrt{\sigma_{1}}} \exp \left(-\frac{a^{2}}{2 \sigma_{1}}\right)\left(\frac{1}{\sqrt{\sigma_{2}}}-\frac{1}{\sqrt{\sigma_{1}+\sigma_{2}}} \exp \left(-\frac{a^{2}}{2\left(\sigma_{1}+\sigma_{2}\right)}\right)\right) \tag{23}
\end{equation*}
$$

Proof. Apply Theorem 2 with $\phi=\bar{\delta}_{a}$ and $\theta \equiv 1$, we obtain:

$$
\mathrm{E}\left[\bar{\delta}_{a}(X)\right]=\frac{1}{\sqrt{2 \pi \sigma_{1}}} \int_{\star_{\mathbb{R}}} \delta_{a}(x) \exp \left(-\frac{x^{2}}{2 \sigma_{1}}\right) d x \approx \frac{1}{\sqrt{2 \pi \sigma_{1}}} \exp \left(-\frac{a^{2}}{2 \sigma_{1}}\right)
$$

Similarly, we have

$$
\mathrm{E}\left[\bar{\delta}_{b}(X+Y)\right] \approx \frac{1}{\sqrt{2 \pi\left(\sigma_{1}+\sigma_{2}\right)}} \exp \left(-\frac{b^{2}}{2\left(\sigma_{1}+\sigma_{2}\right)}\right)
$$

Since $\operatorname{Cov}\left[\bar{\delta}_{a}(X), \bar{\delta}_{b}(X+Y)\right]=\mathrm{E}\left[\bar{\delta}_{a}(X) \bar{\delta}_{b}(X+Y)\right]-\mathrm{E}\left[\bar{\delta}_{a}(X)\right] \mathrm{E}\left[\bar{\delta}_{b}(X+Y)\right]$, the result follows from Lemma 10.

Now we let $\bar{B}_{t}(\omega)=\sum_{0 \leq r<t} \omega_{r}$ be the lifting of Brownian motion as given in the appendix.
Corollary 13. Let $s, t \in \mathbf{T}$, with $0<{ }^{\circ} t<{ }^{\circ} s<\infty$, and $a \in{ }^{*} \mathbb{R}$ be finite, then

$$
\begin{equation*}
\operatorname{Cov}\left[\bar{\delta}_{a}\left(B_{t}\right), \bar{\delta}_{a}\left(B_{s}\right)\right] \approx \frac{1}{2 \pi \sqrt{t}} \exp \left(-\frac{a^{2}}{2 t}\right)\left(\frac{1}{\sqrt{s-t}}-\frac{1}{\sqrt{s}} \exp \left(-\frac{a^{2}}{2 s}\right)\right) \tag{24}
\end{equation*}
$$

Proof. We apply the previous theorem with $\sigma_{1}=t$ and $\sigma_{2}=s-t$.

Corollary 14. Let $X, Y$ be as in the Theorem. Let $K \in{ }^{*} \mathbb{N}$ so that $\bar{\mu}(\Lambda) \approx 1$, where $\Lambda=\{|X+Y| \leq$ $K\}$. Then we can find $\epsilon \approx 0$ small enough in the definition of $\bar{\delta}$ so that

$$
\begin{equation*}
\mathrm{E}_{\Lambda}\left[\bar{\delta}_{X}(X+Y)\right]=\mathrm{E}_{\Lambda}\left[\bar{\delta}_{X+Y}(X)\right] \approx \frac{1}{\sqrt{2 \pi \sigma_{2}}} \tag{25}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{E}_{\Omega \backslash \Lambda}\left[\int_{* \mathbb{R}} \bar{\delta}_{r}(X) \bar{\delta}_{r}(X+Y) d r\right] \approx 0 \tag{26}
\end{equation*}
$$

Note that the result is independent of $X$ and depends only on the variance $\sigma_{2}$ of $Y$.

Proof. Given an $f \in{ }^{*} \mathcal{S}(\mathbb{R})$ and an internal $Z: \bar{\Omega} \rightarrow{ }^{*} \mathbb{R}$ which is bounded a.e. by $K$ in absolute value, we can find for the definition of $\bar{\delta}$ a number $\epsilon \approx 0$ small enough so that $\int_{* \mathbb{R}} f(r) \bar{\delta}_{r}(Z) d r \approx f(Z)$ a.e. Now applying this with $Z=X+Y$ and $f(r)=\bar{\delta}_{r}(X+Y)$, one obtains the corresponding $\epsilon$ and $\Lambda$.

The first equality results from the definition of $\bar{\delta}$ as follows:

$$
\begin{equation*}
\bar{\delta}_{X+Y}(X)=\frac{1}{\sqrt{2 \pi \epsilon}} \exp \left(-\frac{1}{2 \epsilon} Y^{2}\right)=\bar{\delta}_{X}(X+Y) \tag{27}
\end{equation*}
$$

For the rest, we have:

$$
\begin{aligned}
& \int_{* \mathbb{R}} \mathrm{E}\left[\bar{\delta}_{r}(X) \bar{\delta}_{r}(X+Y)\right] d r=\mathrm{E}\left[\int_{* \mathbb{R}} \bar{\delta}_{r}(X) \bar{\delta}_{r}(X+Y) d r\right] \\
\approx & \mathrm{E}_{\Lambda}\left[\bar{\delta}_{X}(X+Y)\right]+\mathrm{E}_{\bar{\Omega} \backslash \Lambda}\left[\int_{* \mathbb{R}} \bar{\delta}_{r}(X) \bar{\delta}_{r}(X+Y) d r\right] \approx \frac{1}{\sqrt{2 \pi \sigma_{2}}}+\mathrm{E}_{\bar{\Omega} \backslash \Lambda}\left[\int_{* \mathbb{R}} \bar{\delta}_{r}(X) \bar{\delta}_{r}(X+Y) d r\right] .
\end{aligned}
$$

On the other hand, the Lemma gives

$$
\int_{* \mathbb{R}} \mathrm{E}\left[\bar{\delta}_{r}(X) \bar{\delta}_{r}(X+Y)\right] d r=\int_{*_{\mathbb{R}}} \frac{1}{2 \pi \sqrt{\sigma_{1} \sigma_{2}}} \exp \left(-\frac{r^{2}}{2 \sigma_{1}}\right) d r=\frac{1}{\sqrt{2 \pi \sigma_{2}}}
$$

If one uses the Loeb measure of $\bar{\mu}$, the above can be stated using finiteness and a Loeb measurable set instead of the bound $K$ and $\Lambda$.

One may think of $\mathrm{E}\left[\bar{\delta}_{X}(X+Y)\right]$ roughly as $\mathrm{E}_{Y=0}[X / \nu]$ for some $\nu \approx 0$.

## 3. A discussion of correlation between Donsker's $\delta$-functions

3.1. In the classical space. We concentrate in this section only on $\delta_{0}\left(B_{t}\right)$ and introduce the notion of $\lambda$-correlation between $\delta_{0}\left(B_{t}\right)$ and $\delta_{0}\left(B_{s}\right)$ as follows:

Definition 15. Let $\delta>0$. The $\lambda$-correlation between $\delta_{0}\left(B_{t}\right)$ and $\delta_{0}\left(B_{s}\right)$ is defined as

$$
\begin{equation*}
\rho_{\lambda}(t, s)=\frac{\operatorname{Cov}_{\lambda}\left[\delta_{0}\left(B_{t}\right), \delta_{0}\left(B_{s}\right)\right]}{\sigma_{\lambda}(t) \sigma_{\lambda}(s)} \tag{28}
\end{equation*}
$$

where

$$
\begin{equation*}
\sigma_{\lambda}^{2}(t)=\operatorname{Cov}_{\lambda}\left[\delta_{0}\left(B_{t}\right), \delta_{0}\left(B_{t}\right)\right] \tag{29}
\end{equation*}
$$

Remark 16. Note that since $\delta_{0}\left(B_{t}\right) \in \mathcal{G}_{-\lambda}$ for every $\lambda>0$, the $\lambda$-variance $\sigma_{\lambda}(t)^{2}$ of $\delta_{0}\left(B_{t}\right)$ is finite. Thus, $\rho_{\lambda}(t, s)$ is a well-defined number. As a matter of fact, it is the (classical) correlation between the $\left(L^{2}\right)$-processes $e^{-\lambda N} \delta_{0}\left(B_{t}\right)$ and $e^{-\lambda N} \delta_{0}\left(B_{s}\right)$.

The next theorem states the expression for the $\lambda$-correlation:
Theorem 17. Let $t \neq s$ and $t, s>0$. Then for any $\lambda>0$

$$
\begin{equation*}
\rho_{\lambda}(t, s)=\sqrt{\frac{1-e^{-4 \lambda}}{1-a(s, t) e^{-4 \lambda}}} \cdot \frac{1-\sqrt{1-a(s, t) e^{-4 \lambda}}}{1-\sqrt{1-e^{-4 \lambda}}} \tag{30}
\end{equation*}
$$

where $a(s, t)=\frac{(t \wedge s)^{2}}{t s}$.
Proof. Observe that

$$
\sigma_{\lambda}^{2}(t)=\frac{1}{2 \pi t}\left(\frac{e^{\lambda}}{\sqrt{e^{2 \lambda}-e^{-2 \lambda}}}-1\right)
$$

By rearranging in the definition of $\lambda$-correlation,

$$
\rho_{\lambda}(t, s)=\frac{e^{\lambda} \sqrt{t s} \sqrt{e^{2 \lambda}-e^{-2 \lambda}}-\sqrt{t s e^{2 \lambda}-(t \wedge s)^{2} e^{-2 \lambda}} \sqrt{e^{2 \lambda}-e^{-2 \lambda}}}{e^{\lambda} \sqrt{t s e^{2 \lambda}-(t \wedge s)^{2} e^{-2 \lambda}}-\sqrt{t s e^{2 \lambda}-(t \wedge s)^{2} e^{-2 \lambda}} \sqrt{e^{2 \lambda}-e^{-2 \lambda}}}
$$

The theorem then follows by simple factorizations.
We study the $\lambda$-correlation as a function of $\lambda$,
Theorem 18. The mapping $\lambda \rightarrow \rho_{\lambda}(t, s)$ is a continuous and monotonically increasing function on $(0, \infty)$ with

$$
\begin{equation*}
\lim _{\lambda \downarrow 0} \rho_{\lambda}(t, s)=0, \quad \lim _{\lambda \uparrow \infty} \rho_{\lambda}(t, s)=\frac{(t \wedge s)^{2}}{t s} \tag{31}
\end{equation*}
$$

Proof. From the expression of $\rho_{\lambda}(t, s)$ in Theorem 17 we see that it is monotonically increasing and continuous as a function of $\lambda$. Moreover, since $\sigma_{\lambda}(t)$ tends to infinity when $\lambda$ goes to zero and the $\lambda$-covariance converges to the generalized covariance, we get $\lim _{\lambda \downarrow 0} \rho_{\lambda}(t, s)=0$. The second limit follows by an application of L'Hôspital's rule.

In analogy with the definition of generalized covariance above, we could introduce the generalized correlation by taking the limit of $\rho_{\lambda}(t, s)$ as $\lambda \downarrow 0$. But, as the theorem says, this number will be 0 . An alternative could be to define the generalized correlation as

$$
\begin{equation*}
\rho(t, s)=\sup _{\lambda>0} \rho_{\lambda}(t, s) \tag{32}
\end{equation*}
$$

Theorem 18 then immediately gives $\rho=t / s$ when $0<t<s$. Compare this result with the correlation between Brownian motion at times $0<t<s$ known to be $\sqrt{t / s}$. Note that we could equivalently have used sup ${ }_{\lambda>0}$ instead of $\lim _{\lambda \downarrow 0}$ in the definition of the generalized covariance.
3.2. In the nonstandard space. Now we define in the nonstandard space a counterpart to the correlation introduced in the last subsection and consider its meaning.

Let $s, t \in T$, with $\bar{\delta}_{0}$ as in $\S 2.2$, then the correlation between $\bar{\delta}_{0}\left(\bar{B}_{t}\right)$ and $\bar{\delta}_{0}\left(\bar{B}_{s}\right)$ is given by

$$
\begin{equation*}
\mathrm{R}\left[\bar{\delta}_{0}\left(\bar{B}_{t}\right), \bar{\delta}_{0}\left(\bar{B}_{s}\right)\right]:=\frac{\operatorname{Cov}\left[\bar{\delta}_{0}\left(\bar{B}_{t}\right), \bar{\delta}_{0}\left(\bar{B}_{s}\right)\right]}{\sqrt{\operatorname{Cov}\left[\bar{\delta}_{0}\left(\bar{B}_{t}\right), \bar{\delta}_{0}\left(\bar{B}_{t}\right)\right] \operatorname{Cov}\left[\bar{\delta}_{0}\left(\bar{B}_{s}\right), \bar{\delta}_{0}\left(\bar{B}_{s}\right)\right]}} . \tag{33}
\end{equation*}
$$

Using equation (22) and the following for $r=s, t$

$$
\begin{equation*}
\mathrm{E}\left[\bar{\delta}_{0}^{2}\left(\bar{B}_{r}\right)\right]=\frac{1}{2 \pi \sqrt{\epsilon(2 r+\epsilon)}}, \quad \mathrm{E}\left[\bar{\delta}_{0}\left(\bar{B}_{r}\right)\right]=\frac{1}{\sqrt{2 \pi(r+\epsilon)}}, \tag{34}
\end{equation*}
$$

the above correlation can be computed as:
$\frac{t^{2} \sqrt{\epsilon \sqrt{2 s+\epsilon} \sqrt{2 t+\epsilon}}}{\sqrt{\left((s-t) t+(s+t) \epsilon+\epsilon^{2}\right)\left(s+\epsilon-\sqrt{2 s \epsilon+\epsilon^{2}}\right)\left(t+\epsilon-\sqrt{2 t \epsilon+\epsilon^{2}}\right)\left(\sqrt{(s+\epsilon)(t+\epsilon)}+\sqrt{(s+\epsilon)(t+\epsilon)-t^{2}}\right)^{2}}}$
If $0<{ }^{\circ} t<{ }^{\circ} s$, the correlation will be infinitesimal, because of $\epsilon \approx 0$. (Compare with the first equation in (31).) However when $\mathrm{R}\left[\bar{\delta}_{0}\left(\bar{B}_{t}\right), \bar{\delta}_{0}\left(\bar{B}_{s}\right)\right]$ is scaled by $\sqrt{\epsilon}$, one obtains:

$$
\begin{equation*}
\frac{1}{\sqrt{\epsilon}} \mathrm{R}\left[\bar{\delta}_{0}\left(\bar{B}_{t}\right), \bar{\delta}_{0}\left(\bar{B}_{s}\right)\right] \approx \frac{t \sqrt{2}}{\sqrt{(s-t) \sqrt{s t}(2 s-t+2 \sqrt{s(s-t)})}} \tag{36}
\end{equation*}
$$

and this is one candidate for the definition of the correlation between $\bar{\delta}_{0}\left(\bar{B}_{t}\right)$ and $\bar{\delta}_{0}\left(\bar{B}_{s}\right)$.
Another possibility is suggested by the second equation in (31). We let $\alpha$ be a positive infinitesimal and scale the Dirac point measure as

$$
\begin{equation*}
\hat{\delta}_{0}(x):=\sqrt{\epsilon} \alpha \bar{\delta}_{0}(\sqrt{\epsilon} \alpha x), \tag{37}
\end{equation*}
$$

then by replacing $\epsilon$ by $1 / \alpha$ in (35) one computes that

$$
\begin{equation*}
\mathrm{R}\left[\hat{\delta}_{0}\left(\bar{B}_{t}\right), \hat{\delta}_{0}\left(\bar{B}_{s}\right)\right] \approx \frac{t}{s} \tag{38}
\end{equation*}
$$

Note that $\hat{\delta}$ is in fact a "flattened" Dirac point measure, a Gaussian density with infinite variance $1 / \alpha$.

## 4. Appendix

Here is a brief introduction to the relevant results from nonstandard analysis. See [1], [10] and [17] for more details.

In the practice of nonstandard analysis, all ordinary mathematical objects $X$, such as sets and functions, are simultaneously extended to new objects ${ }^{*} X$, so that $X$ and ${ }^{*} X$ satisfy the same formal properties which are definable in the first order language of set-theory. When $X$ is infinite the extension * $X$ is proper. We call a subset $Y \subset{ }^{*} X$ internal if $Y$ is defined from ${ }^{*} X$ using first order set-theoretic language. The construction is such that internal sets have the following important compactness property: For any countable family $\mathcal{F}$ of internal subsets of some fixed * $X$, if any finite subfamily of $\mathcal{F}$ has nonempty intersection, then in fact $\cap \mathcal{F} \neq \emptyset$. Internality is seldom explicitly mentioned, unless for emphasis; it is left to the reader to judge from the context.

The existence of such extensions can be proved by methods from logic, such as the ultraproduct construction or Gödel's compactness theorem.

The following is a list of key notions and results that we need.
If $Y$ is an internal set with an internal bijection between $Y$ and $\{1, \ldots, H\}$ for some $H \in{ }^{*} \mathbb{N}$, then $Y$ is called hyperfinite. Let $r, s \in{ }^{*} \mathbb{R}$. If for each $n \in \mathbb{N},|r-s|<1 / n$, then we say that $r, s$ are infinitely close, and write $r \approx s$. When $r \approx 0$, we say that $r$ is an infinitesimal. If for some $n \in \mathbb{N}$,
$|r|<n$, we say that $r$ is finite and write $r<\infty$. For each finite $r \in{ }^{*} \mathbb{R}$, it follows from separation that there is a unique $s \in \mathbb{R}$ such that $r \approx s$. We denote $s={ }^{\circ} r$ and call $s$ the standard part of $r$.

Let $N \in{ }^{*} \mathbb{N} \backslash \mathbb{N}$, i.e. $N$ is infinite but hyperfinite. Let $\Delta t:=1 / N$. Let

$$
\mathbf{T}:=\{-N, \ldots,-\Delta t, 0, \Delta t, \ldots, N\}=\left\{n \Delta t\left|n \in{ }^{*} \mathbb{N},|n| \leq N^{2}\right\}\right.
$$

Then $\mathbf{T}$ is a hyperfinite internal set with internal cardinality $|\mathbf{T}|=N^{2}+1 . \mathbf{T}$ is called the hyperfinite time line - a discrete analog of the real line $(-\infty, \infty)$. Let $\bar{\Omega}:={ }^{*} \mathbb{R}^{\mathbf{T}}$ (so it is $\cong{ }^{*} \mathbb{R}^{N^{2}+1}$ ), equipped with an internal probability measure $\bar{\mu}$ given by $\mathcal{N}(0, \Delta t)^{\mathrm{T}}$. ( $N^{2}+1$ many independent copies.) Write elements of $\bar{\Omega}$ as $\omega=\left(\omega_{t}\right)_{t \in \mathbf{T}}$. So for an internal measurable function $f: \bar{\Omega} \rightarrow{ }^{*} \mathbb{R}$, we have

$$
\mathrm{E}[f]=(2 \pi \Delta t)^{-\frac{N^{2}+1}{2}} \int_{\bar{\Omega}} f(\omega) \exp \left(-\frac{1}{2 \Delta t} \sum_{t \in \mathbf{T}} \omega_{t}^{2}\right) d \omega
$$

In $\bar{\Omega}, \omega_{t}$ models the Brownian increment of the sample path $\omega$ at time $t$, so one can intuitively think of $\omega_{t}$ as $d B_{t}$. The product Gaussian measure on $\bar{\Omega}$ is not $\sigma$-additive. But ${ }^{0} \bar{\mu}$ is an ordinary finitely additive probability measure. (Note that ${ }^{\circ} \bar{\mu}(A)=r$ if $r \in \mathbb{R}$ and $\bar{\mu}(A) \approx r$.) It follows from the above-mentioned compactness and Carathéodory extension that ${ }^{\circ} \bar{\mu}$ extends uniquely to a $\sigma$-additive measure denoted by $\bar{\mu}_{L}$, the Loeb measure of $\bar{\mu}$, after its inventor. Then $\bar{\mu}_{L}$ defines the Wiener measure on $\bar{\Omega}$ and $B{ }_{\circ t}(\omega):={ }^{\circ} \bar{B}_{t}(\omega)$ is the Brownian motion, where $\bar{B}_{t}(\omega):=\sum_{0 \leq r<t} \omega_{r}$. This construction of Brownian motion is due to Anderson [2] (the modification here is due to Cutland [4]), with its motivation from Wiener's classical paper [19].

There is a corresponding integration theory. A function $f: \bar{\Omega} \rightarrow{ }^{*} \mathbb{R}$ is called $S$-integrable (or $S L^{1}$ ) if (i) $\mathrm{E}[|f|]<\infty ;($ ii $) \bar{\mu}(A) \approx 0 \Rightarrow \mathrm{E}_{A}[|f|] \approx 0$. Furthermore, $f$ is called $S L^{p}$ if $f^{p}$ is $S L^{1}$. Each $\bar{\mu}_{L}$-integrable $g: \Omega \rightarrow \mathbb{R}$ has an $S L^{1}$-lifting $f: \Omega \rightarrow \mathbb{R}$ in the sense that $g={ }^{\circ} f$ a.e. $\left(\bar{\mu}_{L}\right)$. Conversely, when $f: \Omega \rightarrow{ }^{*} \mathbb{R}$ is $S L^{1}$, letting $g:={ }^{\circ} f$, then $g$ is $\bar{\mu}$-integrable. From this, we obtain embeddings $L^{2}(W) \hookrightarrow L^{2}(\Omega) \subset{ }^{*} L^{2}(\Omega)$.

See [4],[5], [6], [7] and [11] for more background and applications of $L^{2}(\bar{\Omega})$. See Perkins [14] for an earlier nonstandard approach to local time.

## 5. Epilogue

We hope that in a small way, this North-South / Nonstandard-Standard collaboration reflects and highlights the need for multi-cultural approaches to problems in the world.

## References

[1] S. Albeverio, J.-E. Fenstad, R. Høegh-Krohn and T. Lindstrøm, Nonstandard Methods in Stochastic Analysis and Mathematical Physics, Academic Press, New York, 1986.
[2] R.M. Anderson, A non-standard representation for Brownian motion and Itô integration, Israel J. Math. 25 (1976), 15-46.
[3] F.E. Benth and J. Potthoff, On the martingale property for generalized stochastic processes, Stochastics. 58 (1996), 349-367.
[4] N.J. Cutland, Infinitesimals in action, J. London Math. Soc. 35 (1987), 202-216.
[5] N.J. Cutland and S.-A. Ng, A nonstandard approach to the Malliavin calculus, in "Advances in Analysis, Probability and mathematical Physics - Contributions of Nonstandard Analysis", (edited by Albeverio, Luxemburg, Wolff), Kluwer, 1994, pp149-170.
[6] N.J. Cutland \& S.-A. Ng, On homogeneous chaos, Math. Proc. Camb. Phil. Soc. 110 (1991), 353-363.
[7] N.J. Cutland \& S.-A. Ng, The Wiener sphere and Wiener measure, Ann. of Prob. 21 (1993), 1-13.
[8] T. Hida, H.-H. Kuo, J. Potthoff and L. Streit, White Noise: An infinite dimensional calculus, Kluwer, Dordrecht, 1993.
[9] H. Holden, B. Øksendal, J. Ubøe and T.-S. Zhang, Stochastic Partial Differential Equations - A Modeling, White Noise Functional Approach, Birkhäuser, Boston, 1996.
[10] H.J. Keisler, An infinitesimal approach to stochastic analysis, Mem. Amer. Math. Soc. 297 (1984).
[11] S.-A. Ng, Gradient lines and distributions of functionals in infinite dimensional Euclidean spaces, in "Developments in Nonstandard Mathematics", (edited by Cutland, Neves, Oliveira, Sousa-Pinto), Pitman Series 336, Longman, 1995,186-197.
[12] S.-A. Ng , White noise analysis in a nonstandard model, in preparation, http://www.maths.unp.ac.za/staff/ng/wna.pdf
[13] D. Nualart, The Malliavin Calculus and Related Topics, (Probability and its Applications. A Series of the Applied Probability Trust. Eds.: J. Gani, C.C. Heyde, T.E. Kurtz), Springer-Verlag, Berlin, 1995.
[14] E. Perkins, A global intrinsic characterization of Brownian local time, Ann. of Prob. 9 (1981), 800-817.
[15] J. Potthoff and L. Streit, A characterization of Hida distributions, J. Funct. Anal. 101 (1991), 221-229.
[16] J. Potthoff and M. Timpel, On a dual pair of smooth and generalized random variables, Potential Anal. 4 (1995), 637-654.
[17] A. Robinson, Non-Standard Analysis, McGraw-Hill, New York, 1973.
[18] W. Rudin, Functional Analysis, North-Holland, Amsterdam, 1966.
[19] N. Wiener, Differential space, J. Math. Phys. 2 (1923), 132-174.

Fred Espen Benth
Department of Mathematics
University of Oslo
P.O. Box 1053 Blindern

N-0316 Oslo, Norway
and
MaPhySto - Centre for Mathematical Physics and Stochastics
University of Aarhus
Ny Munkegade
DK-8000 Århus, Denmark
E-mail address: fredb@math.uio.no
Siu-Ah NG
School of Mathematics, Stats \& IT
University of Natal
Private Bag X01
Pietermaritzburg
3209 South Africa
E-mail address: ngs@math.unp.ac.za


[^0]:    give meaning to the covariance of Donsker's delta functions.

